# A first-order augmented Lagrangian method for constrained minimax optimization 

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#### Abstract

In this paper we study a class of constrained minimax problems. In particular, we propose a first-order augmented Lagrangian method for solving them, whose subproblems turn out to be a much simpler structured minimax problem and are suitably solved by a firstorder method recently developed in [26] by the authors. Under some suitable assumptions, an operation complexity of $\mathcal{O}\left(\varepsilon^{-4} \log \varepsilon^{-1}\right)$, measured by its fundamental operations, is established for the first-order augmented Lagrangian method for finding an $\varepsilon$-KKT solution of the constrained minimax problems.


Keywords: minimax optimization, augmented Lagrangian method, first-order method, operation complexity
Mathematics Subject Classification: 90C26, 90C30, 90C47, 90C99, 65 K 05

## 1 Introduction

In this paper, we consider a constrained minimax problem

$$
\begin{equation*}
F^{*}=\min _{c(x) \leq 0} \max _{d(x, y) \leq 0}\{F(x, y):=f(x, y)+p(x)-q(y)\} . \tag{1}
\end{equation*}
$$

Assume that problem (1) has at least one optimal solution and the following additional assumptions hold.

Assumption 1. (i) $F$ is $L_{F}$-Lipschitz continuous on $\mathcal{X} \times \mathcal{Y}, f$ is $L_{\nabla f}$-smooth on $\mathcal{X} \times \mathcal{Y}$, and $f(x, \cdot)$ is concave for any given $x \in \mathcal{X}$, where $\mathcal{X}:=\operatorname{dom} p$ and $\mathcal{Y}:=\operatorname{dom} q \mathbb{1}^{11}$
(ii) $p: \mathbb{R}^{n} \rightarrow \mathbb{R} \cup\{\infty\}$ and $q: \mathbb{R}^{m} \rightarrow \mathbb{R} \cup\{\infty\}$ are proper closed convex functions, and the proximal operator of $p$ and $q$ can be exactly evaluated.
(iii) $c: \mathbb{R}^{n} \rightarrow \mathbb{R}^{\tilde{n}}$ is $L_{\nabla c}$-smooth and $L_{c}$-Lipschitz continuous on $\mathcal{X}$, $d: \mathbb{R}^{n} \times \mathbb{R}^{m} \rightarrow \mathbb{R}^{\tilde{m}}$ is $L_{\nabla d}$-smooth and $L_{d}$-Lipschitz continuous on $\mathcal{X} \times \mathcal{Y}$, and $d_{i}(x, \cdot)$ is convex for each $x \in \mathcal{X}$.
(iv) The sets $\mathcal{X}$ and $\mathcal{Y}$ (namely, $\operatorname{dom} p$ and $\operatorname{dom} q$ ) are compact.

In the recent years, the minimax problem of a simpler form

$$
\begin{equation*}
\min _{x \in X} \max _{y \in Y} f(x ; y), \tag{2}
\end{equation*}
$$

[^0]where $X$ and $Y$ are a closed set, has received tremendous amount of attention. Indeed, it has found broad applications in many areas, such as adversarial training [16, 29, 40, 45, generative adversarial networks [13, 15, 37, reinforcement learning [8, 11, 31, 34, 41, computational game [1, 35, 42, distributed computing [30, 39, prediction and regression [4, 43, 49, 50], and distributionally robust optimization [12, 38. Numerous methods have been developed for solving (2) with $X$ and $Y$ being a simple closed convex set (e.g., see [6, 18, 19, 22, 23, 25, 28, 33, 47, 51, [52, 54]).

There have also been several studies on some other special cases of problem (11) recently. In particular, two first-order methods, called max-oracle gradient-descent and nested gradient descent/ascent methods, were proposed in 14 for solving (1) with $c(x) \equiv 0$ and $p$ and $q$ being the indicator function of simple compact convex sets $X$ and $Y$ respectively, under the assumption that the function $V(x)=\max _{y \in Y}\{f(x, y): d(x, y) \leq 0\}$ is convex and moreover an optimal Lagrangian multiplier associated with the constraint $d(x, y) \leq 0$ can be computed for each $x \in X$. In addition, a multiplier gradient descent method was proposed in 44 for solving (1) with $c(x) \equiv 0, d(x, y)$ being an affine mapping, and $p$ and $q$ being the indicator function of a simple compact convex set. Also, a proximal gradient multi-step ascent decent method was developed in 9 for (11) with $c(x) \equiv 0, d(x, y)$ being an affine mapping, and $f(x, y)=g(x)+x^{T} A y-h(y)$, under the assumption that $f(x, y)-q(y)$ is strongly concave in $y$. Besides, primal dual alternating proximal gradient methods were proposed in [53 for (1) with $c(x) \equiv 0, d(x, y)$ being an affine mapping, and $\{f(x, y)$ being strongly concave in $y$ or $[q(y) \equiv 0$ and $f(x, y)$ being a linear function in $y]\}$. For these methods, an iteration complexity for finding an approximate stationary point of the aforementioned special minimax problem was established in [9, 14, 53], respectively. Yet, their operation complexity, measured by the amount of fundamental operations such as evaluations of gradient of $f$ and proximal operator of $p$ and $q$, was not studied in these works.

There was no algorithmic development for (11) prior to our work, though optimality conditions of (1) were recently studied in [10. In this paper, we propose a first-order augmented Lagrangian (AL) method for solving (11). Specifically, given an iterate $\left(x^{k}, y^{k}\right)$ and a Lagrangian multiplier estimate $\left(\lambda_{\mathbf{x}}^{k}, \lambda_{\mathbf{y}}^{k}\right)$ at the $k$ th iteration, the next iterate $\left(x^{k+1}, y^{k+1}\right)$ is obtained by finding an approximate stationary point of the AL subproblem

$$
\min _{x} \max _{y} \mathcal{L}\left(x, y, \lambda_{\mathbf{x}}^{k}, \lambda_{\mathbf{y}}^{k} ; \rho_{k}\right)
$$

for some $\rho_{k}>0$ through the use of a first-order method proposed in [26], where $\mathcal{L}$ is the AL function of (11) defined as
$\mathcal{L}\left(x, y, \lambda_{\mathbf{x}}, \lambda_{\mathbf{y}} ; \rho\right)=F(x, y)+\frac{1}{2 \rho}\left(\left\|\left[\lambda_{\mathbf{x}}+\rho c(x)\right]_{+}\right\|^{2}-\left\|\lambda_{\mathbf{x}}\right\|^{2}\right)-\frac{1}{2 \rho}\left(\left\|\left[\lambda_{\mathbf{y}}+\rho d(x, y)\right]_{+}\right\|^{2}-\left\|\lambda_{\mathbf{y}}\right\|^{2}\right)$.
The Lagrangian multiplier estimate is then updated by $\lambda_{\mathbf{x}}^{k+1}=\Pi_{\mathbb{B}_{\Lambda}^{+}}\left(\lambda_{\mathbf{x}}^{k}+\rho_{k} c\left(x^{k+1}\right)\right)$ and $\lambda_{\mathbf{y}}^{k+1}=$ $\left[\lambda_{\mathbf{y}}^{k}+\rho_{k} d\left(x^{k+1}, y^{k+1}\right)\right]_{+}$for some $\Lambda>0$, where $\Pi_{\mathbb{B}_{\Lambda}^{+}}(\cdot)$ and $[\cdot]_{+}$are defined in Section 1.1,

The main contributions of this paper are summarized below.

- We propose a first-order AL method for solving problem (11). To the best of our knowledge, this is the first yet implementable method for solving (1).
- We show that under some suitable assumptions, our first-order AL method enjoys an iteration complexity of $\mathcal{O}\left(\log \varepsilon^{-1}\right)$ and an operation complexity of $\mathcal{O}\left(\varepsilon^{-4} \log \varepsilon^{-1}\right)$, measured by the amount of evaluations of $\nabla f, \nabla c, \nabla d$ and proximal operator of $p$ and $q$, for finding an $\varepsilon$-KKT solution of (1).

The rest of this paper is organized as follows. In Subsection 1.1, we introduce some notation and terminology. In Section 2 we propose a first-order AL method for solving problem (11). In

Section 3, we present complexity results for the proposed method. In Section 4, we provide the proof of the main result.

### 1.1 Notation and terminology

The following notation will be used throughout this paper. Let $\mathbb{R}^{n}$ denote the Euclidean space of dimension $n$ and $\mathbb{R}_{+}^{n}$ denote the nonnegative orthant in $\mathbb{R}^{n}$. The standard inner product, $l_{1}$-norm and Euclidean norm are denoted by $\langle\cdot, \cdot\rangle,\|\cdot\|_{1}$ and $\|\cdot\|$, respectively. For any $\Lambda>0$, let $\mathbb{B}_{\Lambda}^{+}=\{x \geq 0:\|x\| \leq \Lambda\}$, whose dimension is clear from the context. For any $v \in \mathbb{R}^{n}$, let $v_{+}$ denote the nonnegative part of $v$, that is, $\left(v_{+}\right)_{i}=\max \left\{v_{i}, 0\right\}$ for all $i$. Given a point $x$ and a closed set $S$ in $\mathbb{R}^{n}$, let $\operatorname{dist}(x, S)=\min _{x^{\prime} \in S}\left\|x^{\prime}-x\right\|, \Pi_{S}(x)$ denote the Euclidean projection of $x$ onto $S$, and $\mathscr{I}_{S}$ denote the indicator function associated with $S$.

A function or mapping $\phi$ is said to be $L_{\phi}$-Lipschitz continuous on a set $S$ if $\left\|\phi(x)-\phi\left(x^{\prime}\right)\right\| \leq$ $L_{\phi}\left\|x-x^{\prime}\right\|$ for all $x, x^{\prime} \in S$. In addition, it is said to be $L_{\nabla \phi}$-smooth on $S$ if $\left\|\nabla \phi(x)-\nabla \phi\left(x^{\prime}\right)\right\| \leq$ $L_{\nabla \phi}\left\|x-x^{\prime}\right\|$ for all $x, x^{\prime} \in S$. For a closed convex function $p: \mathbb{R}^{n} \rightarrow \mathbb{R} \cup\{\infty\}, 2^{2}$ the proximal operator associated with $p$ is denoted by prox $_{p}$, that is,

$$
\operatorname{prox}_{p}(x)=\arg \min _{x^{\prime} \in \mathbb{R}^{n}}\left\{\frac{1}{2}\left\|x^{\prime}-x\right\|^{2}+p\left(x^{\prime}\right)\right\} \quad \forall x \in \mathbb{R}^{n} .
$$

Given that evaluation of $\operatorname{prox}_{\gamma p}(x)$ is often as cheap as $\operatorname{prox}_{p}(x)$, we count the evaluation of $\operatorname{prox}_{\gamma p}(x)$ as one evaluation of proximal operator of $p$ for any $\gamma>0$ and $x \in \mathbb{R}^{n}$.

For a lower semicontinuous function $\phi: \mathbb{R}^{n} \rightarrow \mathbb{R} \cup\{\infty\}$, its domain is the set $\operatorname{dom} \phi:=$ $\{x \mid \phi(x)<\infty\}$. The upper subderivative of $\phi$ at $x \in \operatorname{dom} \phi$ in a direction $d \in \mathbb{R}^{n}$ is defined by

$$
\phi^{\prime}(x ; d)=\limsup _{\substack{x^{\prime} \rightarrow x, t \downarrow 0}} \inf _{d^{\prime} \rightarrow d} \frac{\phi\left(x^{\prime}+t d^{\prime}\right)-\phi\left(x^{\prime}\right)}{t}
$$

where $t \downarrow 0$ means both $t>0$ and $t \rightarrow 0$, and $x^{\prime} \xrightarrow{\phi} x$ means both $x^{\prime} \rightarrow x$ and $\phi\left(x^{\prime}\right) \rightarrow \phi(x)$. The subdifferential of $\phi$ at $x \in \operatorname{dom} \phi$ is the set

$$
\partial \phi(x)=\left\{s \in \mathbb{R}^{n} \mid s^{T} d \leq \phi^{\prime}(x ; d) \quad \forall d \in \mathbb{R}^{n}\right\}
$$

We use $\partial_{x_{i}} \phi(x)$ to denote the subdifferential with respect to $x_{i}$. In addition, for an upper semicontinuous function $\phi$, its subdifferential is defined as $\partial \phi=-\partial(-\phi)$. If $\phi$ is locally Lipschitz continuous, the above definition of subdifferential coincides with the Clarke subdifferential. Besides, if $\phi$ is convex, it coincides with the ordinary subdifferential for convex functions. Also, if $\phi$ is continuously differentiable at $x$, we simply have $\partial \phi(x)=\{\nabla \phi(x)\}$, where $\nabla \phi(x)$ is the gradient of $\phi$ at $x$. In addition, it is not hard to verify that $\partial\left(\phi_{1}+\phi_{2}\right)(x)=\nabla \phi_{1}(x)+\partial \phi_{2}(x)$ if $\phi_{1}$ is continuously differentiable at $x$ and $\phi_{2}$ is lower or upper semicontinuous at $x$. See [7, 46] for more details.

Finally, we introduce an (approximate) stationary point (e.g., see [9, 10, 21) for a general minimax problem

$$
\begin{equation*}
\min _{x} \max _{y} \Psi(x, y), \tag{4}
\end{equation*}
$$

where $\Psi(\cdot, y): \mathbb{R}^{n} \rightarrow \mathbb{R} \cup\{\infty\}$ is a lower semicontinuous function, and $\Psi(x, \cdot): \mathbb{R}^{m} \rightarrow \mathbb{R} \cup\{-\infty\}$ is an upper semicontinuous function.
Definition 1. A point $(x, y)$ is said to be a stationary point of the minimax problem (4) if

$$
0 \in \partial_{x} \Psi(x, y), \quad 0 \in \partial_{y} \Psi(x, y)
$$

In addition, for any $\epsilon>0$, a point $\left(x_{\epsilon}, y_{\epsilon}\right)$ is said to be an $\epsilon$-stationary point of the minimax problem (4) if

$$
\operatorname{dist}\left(0, \partial_{x} \Psi\left(x_{\epsilon}, y_{\epsilon}\right)\right) \leq \epsilon, \quad \operatorname{dist}\left(0, \partial_{y} \Psi\left(x_{\epsilon}, y_{\epsilon}\right)\right) \leq \epsilon
$$

[^1]
## 2 A first-order augmented Lagrangian method for problem (1)

In this section we propose a first-order augmented Lagrangian (FAL) method for problem (11).
One standard approach for solving constrained nonlinear program is to solve a sequence of unconstrained nonlinear program problems, which are typically penalty or augmented Lagrangian subproblems (e.g., see [32]). In a similar spirit, we next propose an FAL method in Algorithm $\mathbb{1}$ for solving (11). In particular, at each iteration, the FAL method finds an approximate stationary point of an AL subproblem in the form of

$$
\begin{equation*}
\min _{x} \max _{y} \mathcal{L}\left(x, y, \lambda_{\mathbf{x}}, \lambda_{\mathbf{y}} ; \rho\right) \tag{5}
\end{equation*}
$$

for some $\rho>0, \lambda_{\mathbf{x}} \in \mathbb{R}_{+}^{\tilde{n}}$ and $\lambda_{\mathbf{y}} \in \mathbb{R}_{+}^{\tilde{m}}$, where $\mathcal{L}$ is the AL function associated with problem (11) defined in (3). In view of Assumption (1) one can observe that $\mathcal{L}$ enjoys the following nice properties.

- For any given $\rho>0, \lambda_{\mathbf{x}} \in \mathbb{R}_{+}^{\tilde{n}}$ and $\lambda_{\mathbf{y}} \in \mathbb{R}_{+}^{\tilde{m}}, \mathcal{L}$ is the sum of smooth function $f(x, y)+$ $\left(\left\|\left[\lambda_{\mathbf{x}}+\rho c(x)\right]_{+}\right\|^{2}-\left\|\lambda_{\mathbf{x}}\right\|^{2}\right) /(2 \rho)-\left(\left\|\left[\lambda_{\mathbf{y}}+\rho d(x, y)\right]_{+}\right\|^{2}-\left\|\lambda_{\mathbf{y}}\right\|^{2}\right) /(2 \rho)$ with Lipschitz continuous gradient and possibly nonsmooth function $p(x)-q(y)$ with exactly computable proximal operator.
- $\mathcal{L}$ is nonconvex in $x$ but concave in $y$.

Thanks to such a nice structure of $\mathcal{L}$, an approximate stationary point of the AL subproblem (55) can be found by Algorithm 3 (see Appendix A), which is a first-order method proposed in [26, Algorithm 2]) for solving nonconvex-concave minimax problems.

Before presenting an FAL method for (1), we let

$$
\begin{align*}
& \mathcal{L}_{\mathbf{x}}\left(x, y, \lambda_{\mathbf{x}} ; \rho\right):=F(x, y)+\frac{1}{2 \rho}\left(\left\|\left[\lambda_{\mathbf{x}}+\rho c(x)\right]_{+}\right\|^{2}-\left\|\lambda_{\mathbf{x}}\right\|^{2}\right),  \tag{6}\\
& c_{\mathrm{hi}}:=\max \{\|c(x)\| x \in \mathcal{X}\}, \quad d_{\mathrm{hi}}:=\max \{\|d(x, y)\| \mid(x, y) \in \mathcal{X} \times \mathcal{Y}\}, \tag{7}
\end{align*}
$$

and make one additional assumption on problem (1).
Assumption 2. For any given $\eta \in(0,1]$, an $\eta$-approximately feasible point $z_{\eta}$ of problem (11), namely $z_{\eta} \in \mathcal{X}$ satisfying $\left\|\left[c\left(z_{\eta}\right)\right]_{+}\right\| \leq \eta$, can be found.

Remark 1. A very similar assumption as Assumption 圆 was considered in [5, 17, [27, 48]. One example of the problem instances satisfying Assumption 圆 arises when the error bound condition $\left.\left\|[c(x)]_{+}\right\|=\mathcal{O}\left(\operatorname{dist}\left(0, \partial\left(\left\|[c(x)]_{+}\right\|^{2}+\mathscr{I}_{\mathcal{X}}(x)\right)\right)\right)^{\nu}\right)$ holds on a level set of $\left\|[c(x)]_{+}\right\|$for some $\nu>0$ (e.g., see [24, 36]). Indeed, one can find the above $z_{\eta}$ by applying a projected gradient method to the problem $\min _{x \in \mathcal{X}}\left\|[c(x)]_{+}\right\|^{2}$.

We are now ready to present an FAL method for solving problem (1).

```
Algorithm 1 A first-order augmented Lagrangian method for problem (1)
    Input: \(\varepsilon, \tau \in(0,1), \epsilon_{0} \in(\tau \varepsilon, 1], \epsilon_{k}=\epsilon_{0} \tau^{k}, \rho_{k}=\epsilon_{k}^{-1}, \Lambda>0, \lambda_{\mathbf{x}}^{0} \in \mathbb{B}_{\Lambda}^{+}, \lambda_{\mathbf{y}}^{0} \in \mathbb{R}_{+}^{\tilde{m}},\left(x^{0}, y^{0}\right) \in\)
        \(\mathcal{X} \times \mathcal{Y}\), and \(x_{\mathbf{n f}} \in \mathcal{X}\) with \(\left\|\left[c\left(x_{\mathbf{n f}}\right)\right]_{+}\right\| \leq \sqrt{\varepsilon}\).
    for \(k=0,1, \ldots\) do
        Set
```

$$
x_{\text {init }}^{k}= \begin{cases}x^{k}, & \text { if } \mathcal{L}_{\mathbf{x}}\left(x^{k}, y^{k}, \lambda_{\mathbf{x}}^{k} ; \rho_{k}\right) \leq \mathcal{L}_{\mathbf{x}}\left(x_{\mathbf{n f}}, y^{k}, \lambda_{\mathbf{x}}^{k} ; \rho_{k}\right)  \tag{8}\\ x_{\mathbf{n f}}, & \text { otherwise }\end{cases}
$$

$3:$
Call Algorithm 3 (see Appendix A) with $\epsilon \leftarrow \epsilon_{k}, \epsilon_{0} \leftarrow \epsilon_{k} /\left(2 \sqrt{\rho_{k}}\right),\left(x^{0}, y^{0}\right) \leftarrow\left(x_{\text {init }}^{k}, y^{k}\right)$ and $L_{\nabla h} \leftarrow L_{k}$ to find an $\epsilon_{k}$-stationary point $\left(x^{k+1}, y^{k+1}\right)$ of

$$
\begin{equation*}
\min _{x} \max _{y} \mathcal{L}\left(x, y, \lambda_{\mathbf{x}}^{k}, \lambda_{\mathbf{y}}^{k} ; \rho_{k}\right) \tag{9}
\end{equation*}
$$

where

$$
\begin{equation*}
L_{k}=L_{\nabla f}+\rho_{k} L_{c}^{2}+\rho_{k} c_{\mathrm{hi}} L_{\nabla c}+\left\|\lambda_{\mathbf{x}}^{k}\right\| L_{\nabla c}+\rho_{k} L_{d}^{2}+\rho_{k} d_{\mathrm{hi}} L_{\nabla d}+\left\|\lambda_{\mathbf{y}}^{k}\right\| L_{\nabla d} \tag{10}
\end{equation*}
$$

Set $\lambda_{\mathbf{x}}^{k+1}=\Pi_{\mathbb{B}_{\Lambda}^{+}}\left(\lambda_{\mathbf{x}}^{k}+\rho_{k} c\left(x^{k+1}\right)\right)$ and $\lambda_{\mathbf{y}}^{k+1}=\left[\lambda_{\mathbf{y}}^{k}+\rho_{k} d\left(x^{k+1}, y^{k+1}\right)\right]_{+}$.
Terminate the algorithm and output $\left(x^{k+1}, y^{k+1}\right)$ if $\epsilon_{k} \leq \varepsilon$.
end for

Remark 2. (i) $x_{\mathbf{n f}}$ is an $\sqrt{\varepsilon}$-approximately feasible point of problem (11), where the subscript "nf" stands for "nearly feasible". It follows from Assumption 圆 that $x_{\mathbf{n f}}$ can be found in advance.
(ii) $\lambda_{\mathbf{x}}^{k+1}$ results from projecting onto a nonnegative Euclidean ball the standard Lagrangian multiplier estimate $\tilde{\lambda}_{\mathbf{x}}^{k+1}$ obtained by the classical scheme $\tilde{\lambda}_{\mathbf{x}}^{k+1}=\left[\lambda_{\mathbf{x}}^{k}+\rho_{k} c\left(x^{k+1}\right)\right]_{+}$. It is called a safeguarded Lagrangian multiplier in the relevant literature [2, 20, 3], which has been shown to enjoy many practical and theoretical advantages (see [2] for discussions).
(iii) In view of Theorem (see Appendix A), one can see that an $\epsilon_{k}$-stationary point of (9) can be successfully found in step 3 of Algorithm 1 by applying Algorithm 3 to problem (9) and thus Algorithm 1 is well-defined.

## 3 Complexity results of Algorithm 1

In this section we establish iteration and operation complexity results for Algorithm 1, Before proceeding, we make one additional assumption that a generalized Mangasarian-Fromowitz constraint qualification holds for the minimization part of (1) and a uniform Slater's condition holds for the maximization part of (1).

Assumption 3. (i) There exist some constants $\delta_{c}, \theta_{a}, \theta_{f}>0$ such that for each $x \in \mathcal{F}\left(\theta_{f}\right)$ there exists some $v_{x} \in \mathbb{R}^{n}$ satisfying $\left\|v_{x}\right\|=1$ and $v_{x}^{T} \nabla c_{i}(x) \leq-\delta_{c}$ for all $i \in \mathcal{A}\left(x ; \theta_{a}\right)$, where

$$
\begin{equation*}
\mathcal{F}\left(\theta_{f}\right)=\left\{x \in \mathcal{X} \mid\left\|[c(x)]_{+}\right\| \leq \theta_{f}\right\}, \quad \mathcal{A}\left(x ; \theta_{a}\right)=\left\{i \mid c_{i}(x) \geq-\theta_{a}, 1 \leq i \leq \tilde{n}\right\} . \tag{11}
\end{equation*}
$$

(ii) For each $x \in \mathcal{X}$, there exists some $\hat{y}_{x} \in \mathcal{Y}$ such that $d_{i}\left(x, \hat{y}_{x}\right)<0$ for all $i=1,2, \ldots, \tilde{m}$, and moreover, $\delta_{d}:=\inf \left\{-d_{i}\left(x, \hat{y}_{x}\right) \mid x \in \mathcal{X}, i=1,2, \ldots, \tilde{m}\right\}>0.3$

[^2]In order to characterize the approximate solution found by Algorithm we next introduce a terminology called an $\varepsilon$-KKT solution of problem (1).

One can observe from Lemma (iii) that problem (11) is equivalent to

$$
\min _{x, \lambda_{\mathbf{y}}}\left\{\max _{y} F(x, y)-\left\langle\lambda_{\mathbf{y}}, d(x, y)\right\rangle+\mathscr{I}_{\mathbb{R}_{+}^{\tilde{+}}}\left(\lambda_{\mathbf{y}}\right) \mid c(x) \leq 0\right\} .
$$

By this, one can further see that problem (11) is equivalent to

$$
\min _{x, \lambda_{\mathbf{y}}} \max _{\lambda_{\mathbf{x}}}\left\{\max _{y}\left\{F(x, y)-\left\langle\lambda_{\mathbf{y}}, d(x, y)\right\rangle+\mathscr{I}_{\mathbb{R}_{+}^{\tilde{m}}}\left(\lambda_{\mathbf{y}}\right)\right\}+\left\langle\lambda_{\mathbf{x}}, c(x)\right\rangle-\mathscr{I}_{\mathbb{R}_{+}^{\tilde{n}}}\left(\lambda_{\mathbf{x}}\right)\right\},
$$

which is a nonconvex-concave minimax problem

$$
\begin{equation*}
\min _{x, \lambda_{\mathbf{y}}} \max _{y, \lambda_{\mathbf{x}}}\left\{F(x, y)+\left\langle\lambda_{\mathbf{x}}, c(x)\right\rangle-\left\langle\lambda_{\mathbf{y}}, d(x, y)\right\rangle-\mathscr{I}_{\mathbb{R}_{+}^{\tilde{n}}}\left(\lambda_{\mathbf{x}}\right)+\mathscr{I}_{\mathbb{R}_{+}^{\tilde{m}}}\left(\lambda_{\mathbf{y}}\right)\right\} \tag{12}
\end{equation*}
$$

It then follows from Definition 1 (see also [9, Theorem 3]) that $\left(x, y, \lambda_{\mathbf{x}}, \lambda_{\mathbf{y}}\right) \in \mathbb{R}^{n} \times \mathbb{R}^{m} \times \mathbb{R}_{+}^{\tilde{n}} \times \mathbb{R}_{+}^{\tilde{m}}$ is a stationary point of problem (12) if

$$
\begin{align*}
& 0 \in \partial_{x} F(x, y)+\nabla c(x) \lambda_{\mathbf{x}}-\nabla_{x} d(x, y) \lambda_{\mathbf{y}}  \tag{13}\\
& 0 \in \partial_{y} F(x, y)-\nabla_{y} d(x, y) \lambda_{\mathbf{y}}  \tag{14}\\
& c(x) \leq 0, \quad\left\langle\lambda_{\mathbf{x}}, c(x)\right\rangle=0  \tag{15}\\
& d(x, y) \leq 0, \quad\left\langle\lambda_{\mathbf{y}}, d(x, y)\right\rangle=0 \tag{16}
\end{align*}
$$

Based on this observation and the equivalence of (11) and (12), we introduce an (approximate) KKT solution of problem (1) below.

Definition 2. The pair $(x, y)$ is said to be a KKT solution of problem (11) if there exists $\left(\lambda_{\mathbf{x}}, \lambda_{\mathbf{y}}\right) \in \mathbb{R}_{+}^{\tilde{n}} \times \mathbb{R}_{+}^{\tilde{m}}$ such that the conditions (13)-(16) hold. In addition, for any $\varepsilon>0,(x, y)$ is said to be an $\varepsilon-K K T$ point of problem (11) if there exists $\left(\lambda_{\mathbf{x}}, \lambda_{\mathbf{y}}\right) \in \mathbb{R}_{+}^{\tilde{n}} \times \mathbb{R}_{+}^{\tilde{m}}$ such that

$$
\begin{aligned}
& \operatorname{dist}\left(0, \partial_{x} F(x, y)+\nabla c(x) \lambda_{\mathbf{x}}-\nabla_{x} d(x, y) \lambda_{\mathbf{y}}\right) \leq \varepsilon \\
& \operatorname{dist}\left(0, \partial_{y} F(x, y)-\nabla_{y} d(x, y) \lambda_{\mathbf{y}}\right) \leq \varepsilon \\
& \left\|[c(x)]_{+}\right\| \leq \varepsilon, \quad\left|\left\langle\lambda_{\mathbf{x}}, c(x)\right\rangle\right| \leq \varepsilon \\
& \left\|[d(x, y)]_{+}\right\| \leq \varepsilon, \quad\left|\left\langle\lambda_{\mathbf{y}}, d(x, y)\right\rangle\right| \leq \varepsilon
\end{aligned}
$$

To study complexity of Algorithm [1, we define

$$
\begin{align*}
& f^{*}(x):=\max \{F(x, y) \mid d(x, y) \leq 0\},  \tag{17}\\
& f_{\text {low }}^{*}:=\inf \left\{f^{*}(x) \mid x \in \mathcal{X}\right\},  \tag{18}\\
& D_{\mathbf{x}}:=\max \{\|u-v\| \| u, v \in \mathcal{X}\}, \quad D_{\mathbf{y}}:=\max \{\|u-v\| \| u, v \in \mathcal{Y}\},  \tag{19}\\
& F_{\text {hi }}:=\max \{F(x, y) \mid(x, y) \in \mathcal{X} \times \mathcal{Y}\}, \quad F_{\text {low }}:=\min \{F(x, y) \mid(x, y) \in \mathcal{X} \times \mathcal{Y}\},  \tag{20}\\
& r:=2 \delta_{d}^{-1}\left(\epsilon_{0}+L_{F}\right) D_{\mathbf{y}},  \tag{21}\\
& K:=\left\lceil\left(\log \varepsilon-\log \epsilon_{0}\right) / \log \tau\right\rceil_{+}, \quad \mathbb{K}:=\{0,1, \ldots, K+1\}, \tag{22}
\end{align*}
$$

where $L_{F}$ and $\delta_{d}$ are given in Assumptions 1 and 3, and $\epsilon_{0}, \varepsilon$, and $\tau$ are some input parameters of Algorithm [1. For convenience, we define $\mathbb{K}-1=\{k-1 \mid k \in \mathbb{K}\}$. One can observe from Assumption 1 that $D_{\mathbf{x}}, D_{\mathbf{y}}, F_{\text {hi }}$ and $F_{\text {low }}$ are finite. Besides, as will be shown in Lemma 1 $f_{\text {low }}^{*}$ is also finite.

We are now ready to present an iteration and operation complexity of Algorithm 1 for finding an $\mathcal{O}(\varepsilon)$-KKT solution of problem (1), whose proof is deferred to Section 4 .

[^3]Theorem 1. Suppose that Assumptions 1, 园 and 3 hold. Let $\left\{\left(x^{k}, y^{k}, \lambda_{\mathbf{x}}^{k}, \lambda_{\mathbf{y}}^{k}\right)\right\}_{k \in \mathbb{K}}$ be generated by Algorithm 1, $c_{\mathrm{hi}}, d_{\mathrm{hi}}, f_{\text {low }}^{*}, D_{\mathbf{x}}, D_{\mathbf{y}}, F_{\mathrm{hi}}, F_{\text {low }}$ and $K$ be defined in (7), (18), (19), (20) and (22), $L_{F}, L_{\nabla f}, L_{\nabla d}, L_{\nabla c}, L_{c}, L_{\nabla d}, L_{d}$ and $\delta_{d}$ be given in Assumption 11, $\varepsilon, \epsilon_{0}, \tau, \Lambda$ and $\lambda_{\mathbf{y}}^{0}$ be given in Algorithm 1, and

$$
\begin{align*}
\widehat{L}= & L_{\nabla f}+L_{c}^{2}+c_{\mathrm{hi}} L_{\nabla c}+\Lambda L_{\nabla c}+L_{d}^{2}+d_{\mathrm{hi}} L_{\nabla d}+L_{\nabla d} \sqrt{\left\|\lambda_{\mathbf{y}}^{0}\right\|^{2}+\frac{2\left(F_{\mathrm{hi}}-f_{\mathrm{low}}^{*}+D_{\mathbf{y}} \epsilon_{0}\right)}{1-\tau}}  \tag{23}\\
\hat{\alpha}= & \min \left\{1, \sqrt{4 /\left(D_{\mathbf{y}} \widehat{L}\right)}\right\}, \quad \hat{\delta}=\left(2+\hat{\alpha}^{-1}\right) \widehat{L} D_{\mathbf{x}}^{2}+\max \left\{1 / D_{\mathbf{y}}, \widehat{L} / 4\right\} D_{\mathbf{y}}^{2}  \tag{24}\\
\widehat{M}= & 16 \max \left\{1 /\left(2 L_{c}^{2}\right), 4 /\left(\hat{\alpha} L_{c}^{2}\right)\right\}\left[\left(3 \widehat{L}+1 /\left(2 D_{\mathbf{y}}\right)\right)^{2} / \min \left\{L_{c}^{2}, 1 /\left(2 D_{\mathbf{y}}\right)\right\}+3 \widehat{L}+1 /\left(2 D_{\mathbf{y}}\right)\right]^{2} \\
& \times\left(\hat{\delta}+2 \hat{\alpha}^{-1}\left(F_{\mathrm{hi}}-F_{\mathrm{low}}+\frac{\Lambda^{2}}{2}+\frac{3}{2}\left\|\lambda_{\mathbf{y}}^{0}\right\|^{2}+\frac{3\left(F_{\mathrm{hi}}-f_{\mathrm{low}}^{*}+D_{\mathbf{y}} \epsilon_{0}\right)}{1-\tau}+\rho_{k} d_{\mathrm{hi}}^{2}+\frac{D_{\mathbf{y}}}{4}+\widehat{L} D_{\mathbf{x}}^{2}\right)\right) \tag{25}
\end{align*}
$$

$\widehat{T}=\left\lceil 16\left(L_{F} D_{\mathbf{y}}+F_{\mathrm{hi}}-f_{\text {low }}^{*}+\Lambda+\frac{1}{2}\left(\tau^{-1}+\left\|\lambda_{\mathbf{y}}^{0}\right\|^{2}\right)+\frac{F_{\mathrm{hi}}-f_{\mathrm{low}}^{*}+D_{\mathbf{y}} \epsilon_{0}}{1-\tau}+\frac{\Lambda^{2}}{2}+\frac{D_{\mathbf{y}}}{4}\right) \widehat{L}\right.$ $\left.+8\left(1+4 D_{\mathbf{y}}^{2} \widehat{L}^{2}\right)\right]_{+}$,
$\tilde{\lambda}_{\mathbf{x}}^{K+1}=\left[\lambda_{\mathbf{x}}^{K}+c\left(x^{K+1}\right) /\left(\epsilon_{0} \tau^{K}\right)\right]_{+}$.
Suppose that

$$
\begin{align*}
& \varepsilon^{-1} \geq \max \left\{1, \theta_{a}^{-1} \Lambda, \theta_{f}^{-2}\left\{2 L_{F} D_{\mathbf{y}}+2 F_{\mathrm{hi}}-2 f_{\mathrm{low}}^{*}+2 \Lambda+\tau^{-1}+\left\|\lambda_{\mathbf{y}}^{0}\right\|^{2}+\frac{2\left(F_{\mathrm{hi}}-f_{\mathrm{low}}^{*}+D_{\mathbf{y}} \epsilon_{0}\right)}{1-\tau}\right.\right. \\
&\left.\left.+\frac{\epsilon_{0} D_{\mathbf{y}}}{2}+L_{c}^{-2}+4 D_{\mathbf{y}}^{2} \widehat{L}+\Lambda^{2}\right\}, \frac{4\left\|\lambda_{\mathbf{y}}^{0}\right\|^{2}}{\delta_{d}^{2} \tau}+\frac{8\left(F_{\mathrm{hi}}-f_{\mathrm{low}}^{*}+D_{\mathbf{y}} \epsilon_{0}\right)}{\delta_{d}^{2} \tau(1-\tau)}\right\} \tag{28}
\end{align*}
$$

Then the following statements hold.
(i) Algorithm 1 terminates after $K+1$ outer iterations and outputs an approximate stationary point $\left(x^{K+1}, y^{K+1}\right)$ of (11) satisfying

$$
\begin{align*}
& \operatorname{dist}\left(0, \partial_{x} F\left(x^{K+1}, y^{K+1}\right)+\nabla c\left(x^{K+1}\right) \tilde{\lambda}_{x}^{K+1}-\nabla_{x} d\left(x^{K+1}, y^{K+1}\right) \lambda_{\mathbf{y}}^{K+1}\right) \leq \varepsilon,  \tag{29}\\
& \operatorname{dist}\left(0, \partial_{y} F\left(x^{K+1}, y^{K+1}\right)-\nabla_{y} d\left(x^{K+1}, y^{K+1}\right) \lambda_{\mathbf{y}}^{K+1}\right) \leq \varepsilon \text {, }  \tag{30}\\
& \left\|\left[c\left(x^{K+1}\right)\right]_{+}\right\| \leq \varepsilon \delta_{c}^{-1}\left(L_{F}+2 L_{d} \delta_{d}^{-1}\left(\epsilon_{0}+L_{F}\right) D_{\mathbf{y}}+\epsilon_{0}\right),  \tag{31}\\
& \left|\left\langle\tilde{\lambda}_{\mathbf{x}}^{K+1}, c\left(x^{K+1}\right)\right\rangle\right| \leq \varepsilon \delta_{c}^{-1}\left(L_{F}+2 L_{d} \delta_{d}^{-1}\left(\epsilon_{0}+L_{F}\right) D_{\mathbf{y}}+\epsilon_{0}\right) \\
& \times \max \left\{\delta_{c}^{-1}\left(L_{F}+2 L_{d} \delta_{d}^{-1}\left(\epsilon_{0}+L_{F}\right) D_{\mathbf{y}}+\epsilon_{0}\right), \Lambda\right\},  \tag{32}\\
& \left\|\left[d\left(x^{K+1}, y^{K+1}\right)\right]_{+}\right\| \leq 2 \varepsilon \delta_{d}^{-1}\left(\epsilon_{0}+L_{F}\right) D_{\mathbf{y}},  \tag{33}\\
& \left|\left\langle\lambda_{\mathbf{y}}^{K+1}, d\left(x^{K+1}, y^{K+1}\right)\right\rangle\right| \leq 2 \varepsilon \delta_{d}^{-1}\left(\epsilon_{0}+L_{F}\right) D_{\mathbf{y}} \max \left\{2 \delta_{d}^{-1}\left(\epsilon_{0}+L_{F}\right) D_{\mathbf{y}},\left\|\lambda_{\mathbf{y}}^{0}\right\|\right\} \tag{34}
\end{align*}
$$

(ii) The total number of evaluations of $\nabla f, \nabla c, \nabla d$ and proximal operator of $p$ and $q$ performed in Algorithm 1 is at most $N$, respectively, where

$$
\begin{align*}
N= & \left(\left[96 \sqrt{2}\left(1+\left(24 \widehat{L}+4 / D_{\mathbf{y}}\right) / L_{c}^{2}\right)\right]+2\right) \max \left\{2, \sqrt{D_{\mathbf{y}} \widehat{L}}\right\} \widehat{T}\left(1-\tau^{4}\right)^{-1} \\
& \times(\tau \varepsilon)^{-4}\left(28 K \log (1 / \tau)+28 \log \left(1 / \epsilon_{0}\right)+2(\log \widehat{M})_{+}+2+2 \log (2 \widehat{T})\right) \tag{35}
\end{align*}
$$

Remark 3. One can observe from Theorem 1 that Algorithm 1 enjoys an iteration complexity of $\mathcal{O}\left(\log \varepsilon^{-1}\right)$ and an operation complexity of $\mathcal{O}\left(\varepsilon^{-4} \log \varepsilon^{-1}\right)$, measured by the amount of evaluations of $\nabla f, \nabla c, \nabla d$ and proximal operator of $p$ and $q$, for finding an $\mathcal{O}(\varepsilon)-K K T$ solution $\left(x^{K+1}, y^{K+1}\right)$ of (11) such that

$$
\begin{aligned}
& \operatorname{dist}\left(\partial_{x} F\left(x^{K+1}, y^{K+1}\right)+\nabla c\left(x^{K+1}\right) \tilde{\lambda}_{\mathbf{x}}-\nabla_{x} d\left(x^{K+1}, y^{K+1}\right) \lambda_{\mathbf{y}}^{K+1}\right) \leq \varepsilon \\
& \operatorname{dist}\left(\partial_{y} F\left(x^{K+1}, y^{K+1}\right)-\nabla_{y} d\left(x^{K+1}, y^{K+1}\right) \lambda_{\mathbf{y}}^{K+1}\right) \leq \varepsilon \\
& \left\|\left[c\left(x^{K+1}\right)\right]_{+}\right\|=\mathcal{O}(\varepsilon), \quad\left|\left\langle\tilde{\lambda}_{\mathbf{x}}^{K+1}, c\left(x^{K+1}\right)\right\rangle\right|=\mathcal{O}(\varepsilon) \\
& \left\|\left[d\left(x^{K+1}, y^{K+1}\right)\right]_{+}\right\|=\mathcal{O}(\varepsilon), \quad\left|\left\langle\lambda_{\mathbf{y}}^{K+1}, d\left(x^{K+1}, y^{K+1}\right)\right\rangle\right|=\mathcal{O}(\varepsilon)
\end{aligned}
$$

where $\tilde{\lambda}_{\mathbf{x}}^{K+1} \in \mathbb{R}_{+}^{\tilde{n}}$ is defined in (27) and $\lambda_{\mathbf{y}}^{K+1} \in \mathbb{R}_{+}^{\tilde{m}}$ is given in Algorithm 1 .

## 4 Proof of the main result

In this section, we provide a proof of our main result presented in Section 2, which is particularly Theorem 1. Before proceeding, let

$$
\begin{equation*}
\mathcal{L}_{\mathbf{y}}\left(x, y, \lambda_{\mathbf{y}} ; \rho\right)=F(x, y)-\frac{1}{2 \rho}\left(\left\|\left[\lambda_{\mathbf{y}}+\rho d(x, y)\right]_{+}\right\|^{2}-\left\|\lambda_{\mathbf{y}}\right\|^{2}\right) \tag{36}
\end{equation*}
$$

In view of (3), (17) and (36), one can observe that

$$
\begin{equation*}
f^{*}(x) \leq \max _{y} \mathcal{L}_{\mathbf{y}}\left(x, y, \lambda_{\mathbf{y}} ; \rho\right) \quad \forall x \in \mathcal{X}, \lambda_{\mathbf{y}} \in \mathbb{R}_{+}^{\tilde{m}}, \rho>0 \tag{37}
\end{equation*}
$$

which will be frequently used later.
We next establish several lemmas that will be used to prove Theorem 1 subsequently.
Lemma 1. Suppose that Assumptions 1 and 3 hold. Let $f^{*}, f_{\text {low }}^{*}, D_{\mathbf{y}}, r, L_{F}$ and $\delta_{d}$ be given in (17), (18), (19), (21) and Assumption 1, respectively. Then the following statements hold.
(i) $\left\|\lambda_{\mathbf{y}}^{*}\right\| \leq \delta_{d}^{-1} L_{F} D_{\mathbf{y}}$ and $\lambda_{\mathbf{y}}^{*} \in \mathbb{B}_{r}^{+}$for all $\lambda_{\mathbf{y}}^{*} \in \Lambda^{*}(x)$ and $x \in \mathcal{X}$, where $\Lambda^{*}(x)$ denotes the set of optimal Lagrangian multipliers of problem (17) for any $x \in \mathcal{X}$.
(ii) The function $f^{*}$ is Lipschitz continuous on $\mathcal{X}$ and $f_{\text {low }}^{*}$ is finite.
(iii) It holds that

$$
\begin{equation*}
f^{*}(x)=\min _{\lambda_{\mathbf{y}}} \max _{y} F(x, y)-\left\langle\lambda_{\mathbf{y}}, d(x, y)\right\rangle+\mathscr{I}_{\mathbb{R}_{+}^{\tilde{m}}}\left(\lambda_{\mathbf{y}}\right) \quad \forall x \in \mathcal{X} \tag{38}
\end{equation*}
$$

where $\mathscr{I}_{\mathbb{R}_{+}^{\tilde{m}}}(\cdot)$ is the indicator function associated with $\mathbb{R}_{+}^{\tilde{m}}$.
Proof. (i) Let $x \in \mathcal{X}$ and $\lambda_{\mathbf{y}}^{*} \in \Lambda^{*}(x)$ be arbitrarily chosen, and let $y^{*} \in \mathcal{Y}$ be such that $\left(y^{*}, \lambda_{\mathbf{y}}^{*}\right)$ is a pair of primal-dual optimal solutions of (17). It then follows that

$$
y^{*} \in \underset{y}{\operatorname{Argmax}} F(x, y)-\left\langle\lambda_{\mathbf{y}}^{*}, d(x, y)\right\rangle, \quad\left\langle\lambda_{\mathbf{y}}^{*}, d\left(x, y^{*}\right)\right\rangle=0, \quad d\left(x, y^{*}\right) \leq 0, \quad \lambda_{\mathbf{y}}^{*} \geq 0
$$

The first relation above yields

$$
F\left(x, y^{*}\right)-\left\langle\lambda_{\mathbf{y}}^{*}, d\left(x, y^{*}\right)\right\rangle \geq F\left(x, \hat{y}_{x}\right)-\left\langle\lambda_{\mathbf{y}}^{*}, d\left(x, \hat{y}_{x}\right)\right\rangle
$$

where $\hat{y}_{x}$ is given in Assumption 3(ii). By this and $\left\langle\lambda_{\mathbf{y}}^{*}, d\left(x, y^{*}\right)\right\rangle=0$, one has

$$
\left\langle\lambda_{\mathbf{y}}^{*},-d\left(x, \hat{y}_{x}\right)\right\rangle \leq F\left(x, y^{*}\right)-F\left(x, \hat{y}_{x}\right)
$$

which together with (19), $\lambda_{\mathbf{y}}^{*} \geq 0$ and Assumption 1 implies that

$$
\begin{equation*}
\delta_{d}\left\|\lambda_{\mathbf{y}}^{*}\right\|_{1} \leq\left\langle\lambda_{\mathbf{y}}^{*},-d\left(x, \hat{y}_{x}\right)\right\rangle \leq F\left(x, y^{*}\right)-F\left(x, \hat{y}_{x}\right) \leq L_{F}\left\|y^{*}-\hat{y}_{x}\right\| \leq L_{F} D_{\mathbf{y}}, \tag{39}
\end{equation*}
$$

where the first inequality is due to Assumption [3(ii), and the third inequality follows from (19) and $L_{F}$-Lipschitz continuity of $F$ (see Assumption $1(\mathrm{i})$ ). Using (21) and (39), we have $\left\|\lambda_{\mathbf{y}}^{*}\right\| \leq\left\|\lambda_{\mathbf{y}}^{*}\right\|_{1} \leq \delta_{d}^{-1} L_{F} D_{\mathbf{y}}$ and hence $\lambda_{\mathbf{y}}^{*} \in \mathbb{B}_{r}^{+}$due to (21).
(ii) Recall from Assumption 1 that $F(x, \cdot)$ and $d_{i}(x, \cdot), i=1, \ldots, l$, are convex for any given $x \in \mathcal{X}$. Using this, (17), (21) and the first statement of this lemma, we observe that

$$
\begin{equation*}
f^{*}(x)=\max _{y} \min _{\lambda \in \mathbb{B}_{r}^{+}} F(x, y)-\langle\lambda, d(x, y)\rangle \quad \forall x \in \mathcal{X} . \tag{40}
\end{equation*}
$$

Notice from Assumption $\mathbb{1}$ that $F$ and $d$ are Lipschitz continuous on their domain. Then it is not hard to observe that $\min \left\{F(x, y)+\langle\lambda, d(x, y)\rangle \mid \lambda \in \mathbb{B}_{r}^{+}\right\}$is a Lipschitz continuous function of $(x, y)$ on its domain. By this and (40), one can easily verify that $f^{*}$ is Lipschitz continuous on $\mathcal{X}$. In addition, the finiteness of $f_{\text {low }}^{*}$ follows from (18), the continuity of $\tilde{f}^{*}$, and the compactness of $\mathcal{X}$.
(iii) One can observe from (17) that for all $x \in \mathcal{X}$,
$f^{*}(x)=\max _{y} \min _{\lambda_{\mathbf{y}}} F(x, y)-\left\langle\lambda_{\mathbf{y}}, d(x, y)\right\rangle+\mathscr{I}_{\mathbb{R}_{+}^{\tilde{m}}}\left(\lambda_{\mathbf{y}}\right) \leq \min _{\lambda_{\mathbf{y}}} \max _{y} F(x, y)-\left\langle\lambda_{\mathbf{y}}, d(x, y)\right\rangle+\mathscr{I}_{\mathbb{R}_{+}^{\tilde{m}}}\left(\lambda_{\mathbf{y}}\right)$, where the inequality follows from the weak duality. In addition, it follows from Assumption 1 that the domain of $F(x, \cdot)$ is compact for all $x \in \mathcal{X}$. By this, (40) and the strong duality, one has

$$
f^{*}(x)=\min _{\lambda \in \mathbb{B}_{r}^{+}} \max _{y} F(x, y)-\langle\lambda, d(x, y)\rangle \quad \forall x \in \mathcal{X},
$$

which together with the above inequality implies that (38) holds.
Lemma 2. Suppose that Assumptions $\mathbb{1}$ and 芧 hold. Let $\left\{\lambda_{\mathbf{y}}^{k}\right\}_{k \in \mathbb{K}}$ be generated by Algorithm $\mathbb{1}$, $f_{\text {low }}^{*}, D_{\mathbf{y}}$, and $F_{\text {hi }}$ be defined in (18), (19) and (20), and $\epsilon_{0}, \tau$, and $\rho_{k}$ be given in Algorithm (1). Then we have

$$
\begin{equation*}
\rho_{k}^{-1}\left\|\lambda_{\mathbf{y}}^{k}\right\|^{2} \leq\left\|\lambda_{\mathbf{y}}^{0}\right\|^{2}+\frac{2\left(F_{\mathrm{hi}}-f_{\text {low }}^{*}+D_{\mathbf{y}} \epsilon_{0}\right)}{1-\tau} \quad \forall 0 \leq k \in \mathbb{K}-1 . \tag{41}
\end{equation*}
$$

Proof. One can observe from (18), (20) and Algorithm 1 that $F_{\text {hi }} \geq f_{\text {low }}^{*}$ and $\rho_{0} \geq 1>\tau>0$, which imply that (41) holds for $k=0$. It remains to show that (41)) holds for all $1 \leq k \in \mathbb{K}-1$.

Since $\left(x^{t+1}, y^{t+1}\right)$ is an $\epsilon_{t}$-stationary point of (9) for all $0 \leq t \in \mathbb{K}-1$, it follows from Definition $\mathbb{\square}$ that there exists some $u \in \partial_{y} \mathcal{L}\left(x^{t+1}, y^{t+1}, \lambda_{\mathbf{x}}^{t}, \lambda_{\mathbf{y}}^{t} ; \rho_{t}, \rho_{t}\right)$ with $\|u\| \leq \epsilon_{t}$. Notice from (3) and (36) that $\partial_{y} \mathcal{L}\left(x^{t+1}, y^{t+1}, \lambda_{\mathbf{x}}^{t}, \lambda_{\mathbf{y}}^{t} ; \rho_{t}, \rho_{t}\right)=\partial_{y} \mathcal{L}_{\mathbf{y}}\left(x^{t+1}, y^{t+1}, \lambda_{\mathbf{y}}^{t} ; \rho_{t}\right)$. Hence, $u \in$ $\partial_{y} \mathcal{L}_{\mathbf{y}}\left(x^{t+1}, y^{t+1}, \lambda_{\mathbf{y}}^{t} ; \rho_{t}\right)$. Also, observe from (1), (36) and Assumption 1 that $\mathcal{L}_{\mathbf{y}}\left(x^{t+1}, \cdot, \lambda_{\mathbf{y}}^{t} ; \rho_{t}\right)$ is concave. Using this, (19), $u \in \partial_{y} \mathcal{L}_{\mathbf{y}}\left(x^{t+1}, y^{t+1}, \lambda_{\mathbf{y}}^{t} ; \rho_{t}\right)$ and $\|u\| \leq \epsilon_{t}$, we obtain

$$
\begin{aligned}
\mathcal{L}_{\mathbf{y}}\left(x^{t+1}, y, \lambda_{\mathbf{y}}^{t} ; \rho_{t}\right) & \leq \mathcal{L}_{\mathbf{y}}\left(x^{t+1}, y^{t+1}, \lambda_{\mathbf{y}}^{t} ; \rho_{t}\right)+\left\langle u, y-y^{t+1}\right\rangle \\
& \leq \mathcal{L}_{\mathbf{y}}\left(x^{t+1}, y^{t+1}, \lambda_{\mathbf{y}}^{t} ; \rho_{t}\right)+D_{\mathbf{y}} \epsilon_{t} \quad \forall y \in \mathcal{Y},
\end{aligned}
$$

which implies that

$$
\begin{equation*}
\max _{y} \mathcal{L}_{\mathbf{y}}\left(x^{t+1}, y, \lambda_{\mathbf{y}}^{t} ; \rho_{t}\right) \leq \mathcal{L}_{\mathbf{y}}\left(x^{t+1}, y^{t+1}, \lambda_{\mathbf{y}}^{t} ; \rho_{t}\right)+D_{\mathbf{y}} \epsilon_{t} . \tag{42}
\end{equation*}
$$

By this, (36) and (37), one has

$$
\begin{aligned}
& f^{*}\left(x^{t+1}\right) \stackrel{\sqrt[37]{3}}{\leq} \max _{y} \mathcal{L}_{\mathbf{y}}\left(x^{t+1}, y, \lambda_{\mathbf{y}}^{t} ; \rho_{t}\right) \\
& \stackrel{\sqrt{366}[42]}{\leq} F\left(x^{t+1}, y^{t+1}\right)-\frac{1}{2 \rho_{t}}\left(\left\|\left[\lambda_{\mathbf{y}}^{t}+\rho_{t} d\left(x^{t+1}, y^{t+1}\right)\right]_{+}\right\|^{2}-\left\|\lambda_{\mathbf{y}}^{t}\right\|^{2}\right)+D_{\mathbf{y}} \epsilon_{t} \\
& \quad=F\left(x^{t+1}, y^{t+1}\right)-\frac{1}{2 \rho_{t}}\left(\left\|\lambda_{\mathbf{y}}^{t+1}\right\|^{2}-\left\|\lambda_{\mathbf{y}}^{t}\right\|^{2}\right)+D_{\mathbf{y}} \epsilon_{t},
\end{aligned}
$$

where the equality follows from the relation $\lambda_{\mathbf{y}}^{t+1}=\left[\lambda_{\mathbf{y}}^{t}+\rho_{t} d\left(x^{t+1}, y^{t+1}\right)\right]_{+}$(see Algorithm (1)). Using the above inequality, (18), (20) and $\epsilon_{t} \leq \epsilon_{0}$ (see Algorithm (1), we have

$$
\left\|\lambda_{\mathbf{y}}^{t+1}\right\|^{2}-\left\|\lambda_{\mathbf{y}}^{t}\right\|^{2} \leq 2 \rho_{k}\left(F\left(x^{t+1}, y^{t+1}\right)-f^{*}\left(x^{t+1}\right)+D_{\mathbf{y}} \epsilon_{t}\right) \leq 2 \rho_{t}\left(F_{\mathrm{hi}}-f_{\mathrm{low}}^{*}+D_{\mathbf{y}} \epsilon_{0}\right) .
$$

Summing up this inequality for $t=0, \ldots, k-1$ with $1 \leq k \in \mathbb{K}-1$ yields

$$
\begin{equation*}
\left\|\lambda_{\mathbf{y}}^{k}\right\|^{2} \leq\left\|\lambda_{\mathbf{y}}^{0}\right\|^{2}+2\left(F_{\mathrm{hi}}-f_{\text {low }}^{*}+D_{\mathbf{y}} \epsilon_{0}\right) \sum_{t=0}^{k-1} \rho_{t} \tag{43}
\end{equation*}
$$

Recall from Algorithm 1 that $\rho_{t}=\epsilon_{t}^{-1}=\left(\epsilon_{0} \tau^{t}\right)^{-1}$. Then we have $\sum_{t=0}^{k-1} \rho_{t} \leq \rho_{k-1} /(1-\tau)$. Using this, (43) and $\rho_{k}>\rho_{k-1} \geq 1$ (see Algorithm (1), we obtain that for all $1 \leq k \in \mathbb{K}-1$,

$$
\rho_{k}^{-1}\left\|\lambda_{\mathbf{y}}^{k}\right\|^{2} \leq \rho_{k}^{-1}\left(\left\|\lambda_{\mathbf{y}}^{0}\right\|^{2}+\frac{2\left(F_{\mathrm{hi}}-f_{\mathrm{low}}^{*}+D_{\mathbf{y}} \epsilon_{0}\right) \rho_{k-1}}{1-\tau}\right) \leq\left\|\lambda_{\mathbf{y}}^{0}\right\|^{2}+\frac{2\left(F_{\mathrm{hi}}-f_{\mathrm{low}}^{*}+D_{\mathbf{y}} \epsilon_{0}\right)}{1-\tau} .
$$

Hence, the conclusion holds as desired.
Lemma 3. Suppose that Assumptions $\square$ and 芧 hold. Let $f_{\text {low }}^{*}, D_{\mathbf{y}}$ and $F_{\text {hi }}$ be defined in (18), (19) and (20), $L_{F}$ and $\delta_{d}$ be given in Assumptions $\mathbb{1}$ and 图, and $\epsilon_{0}, \tau, \epsilon_{k}$ and $\rho_{k}$ be given in Algorithm [1. Suppose that $\left(x^{k+1}, y^{k+1}, \lambda_{\mathbf{y}}^{k+1}\right)$ is generated by Algorithm $\square$ for some $0 \leq k \in \mathbb{K}-1$ with

$$
\begin{equation*}
\rho_{k} \geq \frac{4\left\|\lambda_{\mathbf{y}}^{0}\right\|^{2}}{\delta_{d}^{2}}+\frac{8\left(F_{\mathrm{hi}}-f_{\mathrm{low}}^{*}+D_{\mathbf{y}} \epsilon_{0}\right)}{\delta_{d}^{2}(1-\tau)} . \tag{44}
\end{equation*}
$$

Then we have

$$
\begin{equation*}
\left\|\left[d\left(x^{k+1}, y^{k+1}\right)\right]_{+}\right\| \leq \rho_{k}^{-1}\left\|\lambda_{\mathbf{y}}^{k+1}\right\| \leq 2 \rho_{k}^{-1} \delta_{d}^{-1}\left(\epsilon_{0}+L_{F}\right) D_{\mathbf{y}} . \tag{45}
\end{equation*}
$$

Proof. Suppose that $\left(x^{k+1}, y^{k+1}, \lambda_{\mathbf{y}}^{k+1}\right)$ is generated by Algorithm $\square$ for some $0 \leq k \in \mathbb{K}-1$ with $\rho_{k}$ satisfying (44). Since $\left(x^{k+1}, y^{k+1}\right)$ is an $\epsilon_{k}$-stationary point of (9), it follows from (3) and Definition 1 that

$$
\operatorname{dist}\left(0, \partial_{y} F\left(x^{k+1}, y^{k+1}\right)-\nabla_{y} d\left(x^{k+1}, y^{k+1}\right)\left[\lambda_{\mathbf{y}}^{k}+\rho_{k} d\left(x^{k+1}, y^{k+1}\right)\right]_{+}\right) \leq \epsilon_{k} .
$$

Besides, notice from Algorithm $\square$ that $\lambda_{\mathbf{y}}^{k+1}=\left[\lambda_{\mathbf{y}}^{k}+\rho_{k} d\left(x^{k+1}, y^{k+1}\right)\right]_{+}$. Hence, there exists some $u \in \partial_{y} F\left(x^{k+1}, y^{k+1}\right)$ such that

$$
\begin{equation*}
\left\|u-\nabla_{y} d\left(x^{k+1}, y^{k+1}\right) \lambda_{y}^{k+1}\right\| \leq \epsilon_{k} . \tag{46}
\end{equation*}
$$

By Assumption 3 (ii), there exists some $\hat{y}^{k+1} \in \mathcal{Y}$ such that $-d_{i}\left(x^{k+1}, \hat{y}^{k+1}\right) \geq \delta_{d}$ for all $i$. Notice that $\left\langle\lambda_{\mathbf{y}}^{k+1}, \lambda_{\mathbf{y}}^{k}+\rho_{k} d\left(x^{k+1}, y^{k+1}\right)\right\rangle=\left\|\left[\lambda_{\mathbf{y}}^{k}+\rho_{k} d\left(x^{k+1}, y^{k+1}\right)\right]_{+}\right\|^{2} \geq 0$, which implies that

$$
\begin{equation*}
-\left\langle\lambda_{\mathbf{y}}^{k+1}, \rho_{k}^{-1} \lambda_{\mathbf{y}}^{k}\right\rangle \leq\left\langle\lambda_{\mathbf{y}}^{k+1}, d\left(x^{k+1}, y^{k+1}\right)\right\rangle . \tag{47}
\end{equation*}
$$

Using these and (46), we have

$$
\begin{align*}
& F\left(x^{k+1}, \hat{y}^{k+1}\right)-F\left(x^{k+1}, y^{k+1}\right)+\delta_{d}\left\|\lambda_{\mathbf{y}}^{k+1}\right\|_{1}-\rho_{k}^{-1}\left\langle\lambda_{\mathbf{y}}^{k+1}, \lambda_{\mathbf{y}}^{k}\right\rangle \\
& \leq F\left(x^{k+1}, \hat{y}^{k+1}\right)-F\left(x^{k+1}, y^{k+1}\right)-\left\langle\lambda_{\mathbf{y}}^{k+1}, \rho_{k}^{-1} \lambda_{\mathbf{y}}^{k}+d\left(x^{k+1}, \hat{y}^{k+1}\right)\right\rangle \\
& \left.\stackrel{477}{\leq} F\left(x^{k+1}, \hat{y}^{k+1}\right)-F\left(x^{k+1}, y^{k+1}\right)+\left\langle\lambda_{\mathbf{y}}^{k+1}, d\left(x^{k+1}, y^{k+1}\right)-d\left(x^{k+1}, \hat{y}^{k+1}\right)\right)\right\rangle \\
& \leq\left\langle u, \hat{y}^{k+1}-y^{k+1}\right\rangle+\left\langle\nabla_{y} d\left(x^{k+1}, y^{k+1}\right) \lambda_{\mathbf{y}}^{k+1}, y^{k+1}-\hat{y}^{k+1}\right\rangle \\
& =\left\langle u-\nabla_{y} d\left(x^{k+1}, y^{k+1}\right) \lambda_{\mathbf{y}}^{k+1}, y^{k+1}-\hat{y}^{k+1}\right\rangle \leq D_{\mathbf{y}} \epsilon_{k}, \tag{48}
\end{align*}
$$

where the first inequality is due to $\lambda_{\mathbf{y}}^{k+1} \geq 0$ and $-d_{i}\left(x^{k+1}, \hat{y}^{k+1}\right) \geq \delta_{d}$ for all $i$, the third inequality follows from $u \in \partial_{y} F\left(x^{k+1}, y^{k+1}\right), \lambda_{\mathbf{y}}^{k+1} \geq 0$, the concavity of $F\left(x^{k+1}, \cdot\right)$ and the convexity of $d_{i}\left(x^{k+1}, \cdot\right)$, and the last inequality is due to (19) and (46).

In view of (19), (48) and the Lipschitz continuity of $F$ (see Assumption (1), one has

$$
\begin{align*}
D_{\mathbf{y}} \epsilon_{k}+L_{F} D_{\mathbf{y}} & \stackrel{(19)}{\geq} D_{\mathbf{y}} \epsilon_{k}+L_{F}\left\|\hat{y}^{k+1}-y^{k+1}\right\| \geq D_{\mathbf{y}} \epsilon_{k}-F\left(x^{k+1}, \hat{y}^{k+1}\right)+F\left(x^{k+1}, y^{k+1}\right) \\
& \stackrel{48}{\geq} \delta_{d}\left\|\lambda_{\mathbf{y}}^{k+1}\right\|_{1}-\rho_{k}^{-1}\left\langle\lambda_{\mathbf{y}}^{k+1}, \lambda_{\mathbf{y}}^{k}\right\rangle \geq\left(\delta_{d}-\rho_{k}^{-1}\left\|\lambda_{\mathbf{y}}^{k}\right\|\right)\left\|\lambda_{\mathbf{y}}^{k+1}\right\|, \tag{49}
\end{align*}
$$

where the second inequality follows from $L_{F}$-Lipschitz continuity of $F$, and the last inequality is due to $\left\|\lambda_{\mathbf{y}}^{k+1}\right\|_{1} \geq\left\|\lambda_{\mathbf{y}}^{k+1}\right\|$. In addition, it follows from (41) and (44) that

$$
\delta_{d}-\rho_{k}^{-1}\left\|\lambda_{\mathbf{y}}^{k}\right\| \stackrel{(41)}{\geq} \delta_{d}-\sqrt{\rho_{k}^{-1}\left(\left\|\lambda_{\mathbf{y}}^{0}\right\|^{2}+\frac{2\left(F_{\mathrm{hi}}-f_{\mathrm{low}}^{*}+D_{\mathbf{y}} \epsilon_{0}\right)}{1-\tau}\right)} \stackrel{(44)}{\geq} \frac{1}{2} \delta_{d}
$$

which together with (49) yields

$$
\frac{1}{2} \delta_{d}\left\|\lambda_{\mathbf{y}}^{k+1}\right\| \leq\left(\delta_{d}-\rho_{k}^{-1}\left\|\lambda_{\mathbf{y}}^{k}\right\|\right)\left\|\lambda_{\mathbf{y}}^{k+1}\right\| \stackrel{(49)}{\leq} D_{\mathbf{y}} \epsilon_{k}+L_{F} D_{\mathbf{y}}
$$

The conclusion then follows from this, $\epsilon_{k} \leq \epsilon_{0}$, and the relations

$$
\left\|\left[d\left(x^{k+1}, y^{k+1}\right)\right]_{+}\right\| \leq \rho_{k}^{-1}\left\|\left[\lambda_{\mathbf{y}}^{k}+\rho_{k} d\left(x^{k+1}, y^{k+1}\right)\right]_{+}\right\|=\rho_{k}^{-1}\left\|\lambda_{\mathbf{y}}^{k+1}\right\|
$$

Lemma 4. Suppose that Assumptions 11 and 3 hold. Let $f_{\text {low }}^{*}, D_{\mathbf{y}}$ and $F_{\text {low }}$ be defined in (18), (19) and (20), $L_{F}$ and $\delta_{d}$ be given in Assumptions 1 and 3, $\epsilon_{0}, \tau, \epsilon_{k}, \rho_{k}$ and $\lambda_{\mathbf{y}}^{0}$ be given in Algorithm 1. Suppose that $\left(x^{k+1}, y^{k+1}, \lambda_{\mathbf{x}}^{k+1}, \lambda_{\mathbf{y}}^{k+1}\right)$ is generated by Algorithm 1 for some $0 \leq k \in \mathbb{K}-1$ with

$$
\begin{equation*}
\rho_{k} \geq \frac{4\left\|\lambda_{\mathbf{y}}^{0}\right\|^{2}}{\delta_{d}^{2} \tau}+\frac{8\left(F_{\mathrm{hi}}-f_{\mathrm{low}}^{*}+D_{\mathbf{y}} \epsilon_{0}\right)}{\delta_{d}^{2} \tau(1-\tau)} \tag{50}
\end{equation*}
$$

Let

$$
\begin{equation*}
\tilde{\lambda}_{\mathbf{x}}^{k+1}=\left[\lambda_{\mathbf{x}}^{k}+\rho_{k} c\left(x^{k+1}\right)\right]_{+} . \tag{51}
\end{equation*}
$$

Then we have

$$
\begin{align*}
& \operatorname{dist}\left(0, \partial_{x} F\left(x^{k+1}, y^{k+1}\right)+\nabla c\left(x^{k+1}\right) \tilde{\lambda}_{\mathbf{x}}^{k+1}-\nabla_{x} d\left(x^{k+1}, y^{k+1}\right) \lambda_{\mathbf{y}}^{k+1}\right) \leq \epsilon_{k}  \tag{52}\\
& \operatorname{dist}\left(0, \partial_{y} F\left(x^{k+1}, y^{k+1}\right)-\nabla_{y} d\left(x^{k+1}, y^{k+1}\right) \lambda_{\mathbf{y}}^{k+1}\right) \leq \epsilon_{k}  \tag{53}\\
& \left\|\left[d\left(x^{k+1}, y^{k+1}\right)\right]_{+}\right\| \leq 2 \rho_{k}^{-1} \delta_{d}^{-1}\left(\epsilon_{0}+L_{F}\right) D_{\mathbf{y}}  \tag{54}\\
& \left|\left\langle\lambda_{\mathbf{y}}^{k+1}, d\left(x^{k+1}, y^{k+1}\right)\right\rangle\right| \leq 2 \rho_{k}^{-1} \delta_{d}^{-1}\left(\epsilon_{0}+L_{F}\right) D_{\mathbf{y}} \max \left\{\left\|\lambda_{\mathbf{y}}^{0}\right\|, 2 \delta_{d}^{-1}\left(\epsilon_{0}+L_{F}\right) D_{\mathbf{y}}\right\} . \tag{55}
\end{align*}
$$

Proof. Suppose that $\left(x^{k+1}, y^{k+1}, \lambda_{\mathbf{x}}^{k+1}, \lambda_{\mathbf{y}}^{k+1}\right)$ is generated by Algorithm 1 for some $0 \leq k \in \mathbb{K}-1$ with $\rho_{k}$ satisfying (50). Since $\left(x^{k+1}, y^{k+1}\right)$ is an $\epsilon_{k}$-stationary point of (9), it then follows from Definition 1 that

$$
\begin{equation*}
\operatorname{dist}\left(0, \partial_{x} \mathcal{L}\left(x^{k+1}, y^{k+1}, \lambda_{\mathbf{x}}^{k}, \lambda_{\mathbf{y}}^{k} ; \rho_{k}\right)\right) \leq \epsilon_{k}, \operatorname{dist}\left(0, \partial_{y} \mathcal{L}\left(x^{k+1}, y^{k+1}, \lambda_{\mathbf{x}}^{k}, \lambda_{\mathbf{y}}^{k} ; \rho_{k}\right)\right) \leq \epsilon_{k} \tag{56}
\end{equation*}
$$

Observe from Algorithm 1 that $\lambda_{\mathbf{y}}^{k+1}=\left[\lambda_{\mathbf{y}}^{k}+\rho_{k} d\left(x^{k+1}, y^{k+1}\right)\right]_{+}$. In view of this, (3) and (51), one has

$$
\begin{aligned}
\partial_{x} \mathcal{L}\left(x^{k+1}, y^{k+1}, \lambda_{\mathbf{x}}^{k}, \lambda_{\mathbf{y}}^{k} ; \rho_{k}\right)= & \partial_{x} F\left(x^{k+1}, y^{k+1}\right)+\nabla c\left(x^{k+1}\right)\left[\lambda_{\mathbf{x}}^{k}+\rho_{k} c\left(x^{k+1}\right)\right]_{+} \\
& -\nabla_{x} d\left(x^{k+1}, y^{k+1}\right)\left[\lambda_{\mathbf{y}}^{k}+\rho_{k} d\left(x^{k+1}, y^{k+1}\right)\right]_{+} \\
= & \partial_{x} F\left(x^{k+1}, y^{k+1}\right)+\nabla c\left(x^{k+1}\right) \tilde{\lambda}_{\mathbf{x}}^{k+1}-\nabla_{x} d\left(x^{k+1}, y^{k+1}\right) \lambda_{\mathbf{y}}^{k+1} \\
\partial_{y} \mathcal{L}\left(x^{k+1}, y^{k+1}, \lambda_{\mathbf{x}}^{k}, \lambda_{\mathbf{y}}^{k} ; \rho_{k}\right)= & \partial_{y} F\left(x^{k+1}, y^{k+1}\right)-\nabla_{y} d\left(x^{k+1}, y^{k+1}\right) \lambda_{\mathbf{y}}^{k+1}
\end{aligned}
$$

These relations together with (56) imply that (52) and (53) hold.
Notice from Algorithm 1 that $0<\tau<1$, which together with (50) implies that (44) holds for $\rho_{k}$. It then follows that (45) holds, which immediately yields (54) and

$$
\begin{equation*}
\left\|\lambda_{\mathbf{y}}^{k+1}\right\| \leq 2 \delta_{d}^{-1}\left(\epsilon_{0}+L_{F}\right) D_{\mathbf{y}} \tag{57}
\end{equation*}
$$

Claim that

$$
\begin{equation*}
\left\|\lambda_{\mathbf{y}}^{k}\right\| \leq \max \left\{\left\|\lambda_{\mathbf{y}}^{0}\right\|, 2 \delta_{d}^{-1}\left(\epsilon_{0}+L_{F}\right) D_{\mathbf{y}}\right\} \tag{58}
\end{equation*}
$$

Indeed, (58) clearly holds if $k=0$. We now assume that $k>0$. Notice from Algorithm 1 that $\rho_{k-1}=\tau \rho_{k}$, which together with (50) implies that (44) holds with $k$ replaced by $k-1$. By this and Lemma 3 with $k$ replaced by $k-1$, one can conclude that $\left\|\lambda_{\mathbf{y}}^{k}\right\| \leq 2 \delta_{d}^{-1}\left(\epsilon_{0}+L_{F}\right) D_{\mathbf{y}}$ and hence (58) holds.

We next show that (55) holds. Indeed, by $\lambda_{\mathbf{y}}^{k+1} \geq 0$, (47), (54), (57) and (58), one has

$$
\begin{aligned}
&\left\langle\lambda_{\mathbf{y}}^{k+1}, d\left(x^{k+1}, y^{k+1}\right)\right\rangle \leq\left\langle\lambda_{\mathbf{y}}^{k+1},\left[d\left(x^{k+1}, y^{k+1}\right)\right]_{+}\right\rangle \leq\left\|\lambda_{\mathbf{y}}^{k+1}\right\|\left\|\left[d\left(x^{k+1}, y^{k+1}\right)\right]_{+}\right\| \\
& \stackrel{54)}{\leq} 4 \rho_{k}^{-1} \delta_{d}^{-2}\left(\epsilon_{0}+L_{F}\right)^{2} D_{\mathbf{y}}^{2} \\
&\left\langle\lambda_{\mathbf{y}}^{k+1}, d\left(x^{k+1}, y^{k+1}\right)\right\rangle \stackrel{(47)}{\geq}\left\langle\lambda_{\mathbf{y}}^{k+1},-\rho_{k}^{-1} \lambda_{\mathbf{y}}^{k}\right\rangle \geq-\rho_{k}^{-1}\left\|\lambda_{\mathbf{y}}^{k+1}\right\|\left\|\lambda_{\mathbf{y}}^{k}\right\| \\
& \geq-2 \rho_{k}^{-1} \delta_{d}^{-1}\left(\epsilon_{0}+L_{F}\right) D_{\mathbf{y}} \max \left\{\left\|\lambda_{\mathbf{y}}^{0}\right\|, 2 \delta_{d}^{-1}\left(\epsilon_{0}+L_{F}\right) D_{\mathbf{y}}\right\} .
\end{aligned}
$$

These relations imply that (55) holds.
Lemma 5. Suppose that Assumptions $\mathbb{1}$, 2 and 3 hold. Let $\left\{\left(\lambda_{\mathbf{x}}^{k}, \lambda_{\mathbf{y}}^{k}\right)\right\}_{k \in \mathbb{K}}$ be generated by Algorithm 11, $\mathcal{L}, f_{\text {low }}^{*}, D_{\mathbf{y}}$ and $F_{\mathrm{hi}}$ be defined in (3), (18), (19) and (20), $L_{F}$ be given in Assumption 1. and $\epsilon_{0}, \tau, \rho_{k}, \Lambda$ and $x_{\text {init }}^{k}$ be given in Algorithm 1. Then for all $0 \leq k \in \mathbb{K}-1$, we have

$$
\begin{equation*}
\max _{y} \mathcal{L}\left(x_{\mathrm{init}}^{k}, y, \lambda_{\mathbf{x}}^{k}, \lambda_{\mathbf{y}}^{k} ; \rho_{k}\right) \leq L_{F} D_{\mathbf{y}}+F_{\mathrm{hi}}+\Lambda+\frac{1}{2}\left(\tau^{-1}+\left\|\lambda_{\mathbf{y}}^{0}\right\|^{2}\right)+\frac{F_{\mathrm{hi}}-f_{\mathrm{low}}^{*}+D_{\mathbf{y}} \epsilon_{0}}{1-\tau} \tag{59}
\end{equation*}
$$

Proof. In view of (6), (8), (20) and $\left\|\lambda_{\mathbf{x}}^{k}\right\| \leq \Lambda$ (see Algorithm (1), one has

$$
\begin{align*}
\mathcal{L}_{\mathbf{x}}\left(x_{\mathrm{init}}^{k}, y^{k}, \lambda_{\mathbf{x}}^{k} ; \rho_{k}\right) & \stackrel{(8)}{\leq} \mathcal{L}_{\mathbf{x}}\left(x_{\mathbf{n f}}, y^{k}, \lambda_{\mathbf{x}}^{k} ; \rho_{k}\right) \stackrel{(6)}{=} F\left(x_{\mathbf{n f}}, y^{k}\right)+\frac{1}{2 \rho_{k}}\left(\left\|\left[\lambda_{\mathbf{x}}^{k}+\rho_{k} c\left(x_{\mathbf{n f}}\right)\right]_{+}\right\|^{2}-\left\|\lambda_{\mathbf{x}}^{k}\right\|^{2}\right) \\
& \leq F\left(x_{\mathbf{n f}}, y^{k}\right)+\frac{1}{2 \rho_{k}}\left(\left(\left\|\lambda_{\mathbf{x}}^{k}\right\|+\rho_{k}\left\|\left[c\left(x_{\mathbf{n f}}\right)\right]_{+}\right\|\right)^{2}-\left\|\lambda_{\mathbf{x}}^{k}\right\|^{2}\right) \\
& =F\left(x_{\mathbf{n f}}, y^{k}\right)+\left\|\lambda_{\mathbf{x}}^{k}\right\|\left\|\left[c\left(x_{\mathbf{n f}}\right)\right]_{+}\right\|+\frac{1}{2} \rho_{k}\left\|\left[c\left(x_{\mathbf{n f}}\right)\right]_{+}\right\|^{2} \\
& \stackrel{(20)}{\leq} F_{\mathrm{hi}}+\Lambda\left\|\left[c\left(x_{\mathbf{n f}}\right)\right]_{+}\right\|+\frac{1}{2} \rho_{k}\left\|\left[c\left(x_{\mathbf{n f}}\right)\right]_{+}\right\|^{2} \tag{60}
\end{align*}
$$

In addition, one can observe from Algorithm 1 that $\epsilon_{k}>\tau \varepsilon$ for all $0 \leq k \in \mathbb{K}-1$. By this and the choice of $\rho_{k}$ in Algorithm 1, we obtain that $\rho_{k}=\epsilon_{k}^{-1}<\tau^{-1} \varepsilon^{-1}$ for all $0 \leq k \in \mathbb{K}-1$. It then follows from this, (31), (6), (19), (41), (60), $\left\|\left[c\left(x_{\mathbf{n f}}\right)\right]_{+}\right\| \leq \sqrt{\varepsilon} \leq 1$, and the Lipschitz continuity
of $F$ that

$$
\begin{aligned}
& \max _{y} \mathcal{L}\left(x_{\mathrm{init}}^{k}, y, \lambda_{\mathbf{x}}^{k}, \lambda_{\mathbf{y}}^{k} ; \rho_{k}\right) \stackrel{(3) \sqrt[(6)]{=}}{=} \max _{y}\left\{\mathcal{L}_{\mathbf{x}}\left(x_{\mathrm{init}}^{k}, y, \lambda_{\mathbf{x}}^{k} ; \rho_{k}\right)-\frac{1}{2 \rho_{k}}\left(\left\|\left[\lambda_{\mathbf{y}}^{k}+\rho_{k} d\left(x_{\mathrm{init}}^{k}, y\right)\right]_{+}\right\|^{2}-\left\|\lambda_{\mathbf{y}}^{k}\right\|^{2}\right)\right\} \\
& \leq \max _{y}\left\{\mathcal{L}_{\mathbf{x}}\left(x_{\mathrm{init}}^{k}, y, \lambda_{\mathbf{x}}^{k} ; \rho_{k}\right)+\frac{1}{2 \rho_{k}}\left\|\lambda_{\mathbf{y}}^{k}\right\|^{2}\right\} \\
& \text { (6) } \max _{y}\left\{F\left(x_{\mathrm{init}}^{k}, y\right)-F\left(x_{\mathrm{init}}^{k}, y^{k}\right)+\mathcal{L}_{\mathbf{x}}\left(x_{\mathrm{init}}^{k}, y^{k}, \lambda_{\mathbf{x}}^{k} ; \rho_{k}\right)+\frac{1}{2 \rho_{k}}\left\|\lambda_{\mathbf{y}}^{k}\right\|^{2}\right\} \\
& \leq \max _{y \in \mathcal{Y}} L_{F}\left\|y-y^{k}\right\|+\mathcal{L}_{\mathbf{x}}\left(x_{\mathrm{init}}^{k}, y^{k}, \lambda_{\mathbf{x}}^{k} ; \rho_{k}\right)+\frac{1}{2 \rho_{k}}\left\|\lambda_{\mathbf{y}}^{k}\right\|^{2} \\
& \leq L_{F} D_{\mathbf{y}}+F_{\mathrm{hi}}+\Lambda\left\|\left[c\left(x_{\mathbf{n f}}\right)\right]_{+}\right\|+\frac{1}{2} \rho_{k}\left\|\left[c\left(x_{\mathrm{nf}}\right)\right]_{+}\right\|^{2}+\frac{1}{2}\left\|\lambda_{\mathbf{y}}^{0}\right\|^{2}+\frac{F_{\mathrm{hi}}-f_{\text {low }}^{*}+D_{\mathbf{y}} \epsilon_{0}}{1-\tau} \\
& \leq L_{F} D_{\mathbf{y}}+F_{\mathrm{hi}}+\Lambda+\frac{1}{2}\left(\tau^{-1}+\left\|\lambda_{\mathbf{y}}^{0}\right\|^{2}\right)+\frac{F_{\mathrm{hi}}-f_{\mathrm{low}}^{*}+D_{\mathbf{y}} \epsilon_{0}}{1-\tau},
\end{aligned}
$$

where the second inequality follows from $L_{F}$－Lipschitz continuity of $F$（see Assumption $\rrbracket(i)$ ）， the third inequality follows from（19），（41）and（60），and the last inequality follows from $\rho_{k}<$ $\tau^{-1} \varepsilon^{-1}$ and $\left\|\left[c\left(x_{\mathbf{n f}}\right)\right]_{+}\right\| \leq \sqrt{\varepsilon} \leq 1$ ．

Lemma 6．Suppose that Assumptions园，圆 and 圆hold．Let $L_{k}, f_{\text {low }}^{*}, D_{\mathbf{x}}, D_{\mathbf{y}}, F_{\text {hi }}$ and $F_{\text {low }}$ be defined in（10），（18），（19）and（20），$L_{F}$ be given in Assumption［1，$\epsilon_{0}, \tau, \epsilon_{k}, \rho_{k}, \Lambda$ and $\lambda_{\mathbf{y}}^{0}$ be given in Algorithm 1，and

$$
\begin{align*}
\alpha_{k}= & \min \left\{1, \sqrt{4 \epsilon_{k} /\left(D_{\mathbf{y}} L_{k}\right)}\right\},  \tag{61}\\
\delta_{k}= & \left(2+\alpha_{k}^{-1}\right) L_{k} D_{\mathbf{x}}^{2}+\max \left\{\epsilon_{k} / D_{\mathbf{y}}, \alpha_{k} L_{k} / 4\right\} D_{\mathbf{y}}^{2},  \tag{62}\\
M_{k}= & \frac{16 \max \left\{1 /\left(2 L_{k}\right), \min \left\{D_{\mathbf{y}} / \epsilon_{k}, 4 /\left(\alpha_{k} L_{k}\right)\right\}\right\} \rho_{k}}{\left[\left(3 L_{k}+\epsilon_{k} /\left(2 D_{\mathbf{y}}\right)\right)^{2} / \min \left\{L_{k}, \epsilon_{k} /\left(2 D_{\mathbf{y}}\right)\right\}+3 L_{k}+\epsilon_{k} /\left(2 D_{\mathbf{y}}\right)\right]^{-2} \epsilon_{k}^{2}} \times\left(\delta_{k}+2 \alpha_{k}^{-1}\left(F_{\mathrm{hi}}-F_{\text {low }}\right.\right. \\
& \left.\left.+\frac{\Lambda^{2}}{2 \rho_{k}}+\frac{3}{2}\left\|\lambda_{\mathbf{y}}^{0}\right\|^{2}+\frac{3\left(F_{\mathrm{hi}}-f_{\text {low }}^{*}+D_{\mathbf{y}} \epsilon_{0}\right)}{1-\tau}+\rho_{k} d_{\mathrm{hi}}^{2}+\frac{\epsilon_{k} D_{\mathbf{y}}}{4}+L_{k} D_{\mathbf{x}}^{2}\right)\right)  \tag{63}\\
T_{k}= & {\left[16\left(L_{F} D_{\mathbf{y}}+F_{\mathrm{hi}}-f_{\text {low }}^{*}+\Lambda+\frac{1}{2}\left(\tau^{-1}+\left\|\lambda_{\mathbf{y}}^{0}\right\|^{2}\right)+\frac{F_{\mathrm{hi}}-f_{\text {low }}^{*}+D_{\mathbf{y}} \epsilon_{0}}{1-\tau}+\frac{\Lambda^{2}}{2 \rho_{k}}+\frac{\epsilon_{k} D_{\mathbf{y}}}{4}\right) L_{k} \epsilon_{k}^{-2}\right.} \\
& \left.+8\left(1+4 D_{\mathbf{y}}^{2} L_{k}^{2} \epsilon_{k}^{-2}\right) \rho_{k}^{-1}-1\right\rceil_{+}  \tag{64}\\
N_{k}= & \left(\left[96 \sqrt{2}\left(1+\left(24 L_{k}+4 \epsilon_{k} / D_{\mathbf{y}}\right) L_{k}^{-1}\right)\right\rceil+2\right) \max \left\{2, \sqrt{D_{\mathbf{y}} L_{k} \epsilon_{k}^{-1}}\right\} \\
& \times\left(\left(T_{k}+1\right)\left(\log M_{k}\right)_{+}+T_{k}+1+2 T_{k} \log \left(T_{k}+1\right)\right) . \tag{65}
\end{align*}
$$

Then for all $0 \leq k \in \mathbb{K}-1$ ，Algorithm $⿴ 囗 十$ finds an $\epsilon_{k}$－stationary point $\left(x^{k+1}, y^{k+1}\right)$ of problem （9）that satisfies

$$
\begin{align*}
\max _{y} \mathcal{L}\left(x^{k+1}, y, \lambda_{\mathbf{x}}^{k}, \lambda_{\mathbf{y}}^{k} ; \rho_{k}\right) \leq & L_{F} D_{\mathbf{y}}+F_{\mathrm{hi}}+\Lambda+\frac{1}{2}\left(\tau^{-1}+\left\|\lambda_{\mathbf{y}}^{0}\right\|^{2}\right)+\frac{F_{\mathrm{hi}}-f_{\text {low }}^{*}+D_{\mathbf{y}} \epsilon_{0}}{1-\tau} \\
& +\frac{\epsilon_{k} D_{\mathbf{y}}}{4}+\frac{1}{2 \rho_{k}}\left(L_{k}^{-1} \epsilon_{k}^{2}+4 D_{\mathbf{y}}^{2} L_{k}\right) . \tag{66}
\end{align*}
$$

Moreover，the total number of evaluations of $\nabla f, \nabla c, \nabla d$ and proximal operator of $p$ and $q$ performed in iteration $k$ of Algorithm $\mathbb{1}$ is no more than $N_{k}$ ，respectively．

Proof. Observe from (11) and (3) that problem (9) can be viewed as

$$
\min _{x} \max _{y}\{h(x, y)+p(x)-q(y)\},
$$

where

$$
h(x, y)=f(x, y)+\frac{1}{2 \rho_{k}}\left(\left\|\left[\lambda_{\mathbf{x}}^{k}+\rho_{k} c(x)\right]_{+}\right\|^{2}-\left\|\lambda_{\mathbf{x}}^{k}\right\|^{2}\right)-\frac{1}{2 \rho_{k}}\left(\left\|\left[\lambda_{\mathbf{y}}^{k}+\rho_{k} d(x, y)\right]_{+}\right\|^{2}-\left\|\lambda_{\mathbf{y}}^{k}\right\|^{2}\right) .
$$

Notice that

$$
\begin{aligned}
& \nabla_{x} h(x, y)=\nabla_{x} f(x, y)+\nabla_{c}(x)\left[\lambda_{\mathbf{x}}^{k}+\rho_{k} c(x)\right]_{+}+\nabla_{x} d(x, y)\left[\lambda_{\mathrm{y}}^{k}+\rho_{k} d(x, y)\right]_{+}, \\
& \nabla_{y} h(x, y)=\nabla_{y} f(x, y)+\nabla_{y} d(x, y)\left[\lambda_{\mathbf{y}}^{k}+\rho_{k} d(x, y)\right]_{+}
\end{aligned}
$$

It follows from Assumption (iii) that

$$
\|\nabla c(x)\| \leq L_{c}, \quad\|\nabla d(x, y)\| \leq L_{d} \quad \forall(x, y) \in \mathcal{X} \times \mathcal{Y} .
$$

In view of the above relations, (17) and Assumption (1) one can observe that $\nabla c(x)\left[\lambda_{\mathbf{x}}^{k}+\rho_{k} c(x)\right]_{+}$ is $\left(\rho_{k} L_{c}^{2}+\rho_{k} c_{\mathrm{hi}} L_{\nabla c}+\left\|\lambda_{\mathbf{x}}^{k}\right\| L_{\nabla c}\right)$-Lipschitz continuous on $\mathcal{X}$, and $\nabla d(x, y)\left[\lambda_{\mathbf{y}}^{k}+\rho_{k} d(x, y)\right]_{+}$is $\left(\rho_{k} L_{d}^{2}+\rho_{k} d_{\mathrm{hi}} L_{\nabla d}+\left\|\lambda_{\mathbf{y}}^{k}\right\| L_{\nabla d}\right)$-Lipschitz continuous on $\mathcal{X} \times \mathcal{Y}$. Using these and the fact that $\nabla f(x, y)$ is $L_{\nabla f}$-Lipschitz continuous on $\mathcal{X} \times \mathcal{Y}$, we can see that $h(x, y)$ is $L_{k}$-smooth on $\mathcal{X} \times \mathcal{Y}$ for all $0 \leq k \in \mathbb{K}-1$, where $L_{k}$ is given in (10). Consequently, it follows from Theorem 2 that Algorithm 3 can be suitably applied to problem (9) for finding an $\epsilon_{k}$-stationary point $\left(x^{k+1}, y^{k+1}\right)$ of it.

In addition, by (3), (18), (36), (37) and $\left\|\lambda_{\mathbf{x}}^{k}\right\| \leq \Lambda$ (see Algorithm (1)), one has

$$
\begin{align*}
& \min _{x} \max _{y} \mathcal{L}\left(x, y, \lambda_{\mathbf{x}}^{k}, \lambda_{\mathbf{y}}^{k} ; \rho_{k}\right) \stackrel{(3) \sqrt[(36)]{=}}{=} \min _{x} \max _{y}\left\{\mathcal{L}_{\mathbf{y}}\left(x, y, \lambda_{\mathbf{y}}^{k} ; \rho_{k}\right)+\frac{1}{2 \rho_{k}}\left(\left\|\left[\lambda_{\mathbf{x}}^{k}+\rho_{k} c(x)\right]_{+}\right\|^{2}-\left\|\lambda_{\mathbf{x}}^{k}\right\|^{2}\right)\right\} \\
& \stackrel{(37)}{\geq} \min _{x}\left\{f^{*}(x)+\frac{1}{2 \rho_{k}}\left(\left\|\left[\lambda_{\mathbf{x}}^{k}+\rho_{k} c(x)\right]_{+}\right\|^{2}-\left\|\lambda_{\mathbf{x}}^{k}\right\|^{2}\right)\right\} \stackrel{(18)}{\geq} f_{\text {low }}^{*}-\frac{1}{2 \rho_{k}}\left\|\lambda_{\mathbf{x}}^{k}\right\|^{2} \geq f_{\text {low }}^{*}-\frac{\Lambda^{2}}{2 \rho_{k}} . \tag{67}
\end{align*}
$$

Let $\left(x^{*}, y^{*}\right)$ be an optimal solution of (1). It then follows that $c\left(x^{*}\right) \leq 0$. Using this, (3), (20) and (41), we obtain that

$$
\begin{align*}
& \min _{x} \max _{y} \mathcal{L}\left(x, y, \lambda_{\mathbf{x}}^{k}, \lambda_{\mathbf{y}}^{k} ; \rho_{k}\right) \leq \max _{y} \mathcal{L}\left(x^{*}, y, \lambda_{\mathbf{x}}^{k}, \lambda_{\mathbf{y}}^{k} ; \rho_{k}\right) \\
& \stackrel{(\Omega 3)}{=} \max _{y}\left\{F\left(x^{*}, y\right)+\frac{1}{2 \rho_{k}}\left(\left\|\left[\lambda_{\mathbf{x}}^{k}+\rho_{k} c\left(x^{*}\right)\right]_{+}\right\|^{2}-\left\|\lambda_{\mathbf{x}}^{k}\right\|^{2}\right)-\frac{1}{2 \rho_{k}}\left(\left\|\left[\lambda_{\mathbf{y}}^{k}+\rho_{k} d\left(x^{*}, y\right)\right]_{+}\right\|^{2}-\left\|\lambda_{\mathbf{y}}^{k}\right\|^{2}\right)\right\} \\
& \leq \max _{y}\left\{F\left(x^{*}, y\right)-\frac{1}{2 \rho_{k}}\left(\left\|\left[\lambda_{\mathbf{y}}^{k}+\rho_{k} d\left(x^{*}, y\right)\right]_{+}\right\|^{2}-\left\|\lambda_{\mathbf{y}}^{k}\right\|^{2}\right)\right\} \\
& \leq F_{\mathrm{hi}}+\frac{1}{2 \rho_{k}}\left\|\lambda_{\mathbf{y}}^{k}\right\|^{2} \stackrel{[410}{\leq} F_{\mathrm{hi}}+\frac{1}{2}\left\|\lambda_{\mathbf{y}}^{0}\right\|^{2}+\frac{F_{\mathrm{hi}}-f_{\text {low }}^{*}+D_{\mathbf{y}} \epsilon_{0}}{1-\tau} \tag{68}
\end{align*}
$$

where the second inequality is due to $c\left(x^{*}\right) \leq 0$. Moreover, it follows from this, (3), (77), (201), (41), $\lambda_{\mathbf{y}}^{k} \in \mathbb{R}_{+}^{\tilde{m}}$ and $\left\|\lambda_{\mathbf{x}}^{k}\right\| \leq \Lambda$ that

$$
\begin{align*}
& \min _{(x, y) \in \mathcal{X} \times \mathcal{Y}} \mathcal{L}\left(x, y, \lambda_{\mathbf{x}}^{k}, \lambda_{\mathbf{y}}^{k} ; \rho_{k}\right) \stackrel{(3)}{\geq} \min _{(x, y) \in \mathcal{X} \times \mathcal{Y}}\left\{F(x, y)-\frac{1}{2 \rho_{k}}\left\|\lambda_{\mathbf{x}}^{k}\right\|^{2}-\frac{1}{2 \rho_{k}}\left\|\left[\lambda_{\mathbf{y}}^{k}+\rho_{k} d(x, y)\right]_{+}\right\|^{2}\right\} \\
& \geq \min _{(x, y) \in \mathcal{X} \times \mathcal{Y}}\left\{F(x, y)-\frac{1}{2 \rho_{k}}\left\|\lambda_{\mathbf{x}}^{k}\right\|^{2}-\frac{1}{2 \rho_{k}}\left(\left\|\lambda_{\mathbf{y}}^{k}\right\|+\rho_{k}\left\|[d(x, y)]_{+}\right\|\right)^{2}\right\} \\
& \geq \min _{(x, y) \in \mathcal{X} \times \mathcal{Y}}\left\{F(x, y)-\frac{1}{2 \rho_{k}}\left\|\lambda_{\mathbf{x}}^{k}\right\|^{2}-\rho_{k}^{-1}\left\|\lambda_{\mathbf{y}}^{k}\right\|^{2}-\rho_{k}\left\|[d(x, y)]_{+}\right\|^{2}\right\} \\
& \geq F_{\text {low }}-\frac{\Lambda^{2}}{2 \rho_{k}}-\left\|\lambda_{\mathbf{y}}^{0}\right\|^{2}-\frac{2\left(F_{\mathrm{hi}}-f_{\text {low }}^{*}+D_{\mathbf{y}} \epsilon_{0}\right)}{1-\tau}-\rho_{k} d_{\mathrm{hi}}^{2}, \tag{69}
\end{align*}
$$

where the second inequality is due to $\lambda_{\mathbf{y}}^{k} \in \mathbb{R}_{+}^{\tilde{m}}$ and the last inequality is due to (7), (20), (41) and $\left\|\lambda_{\mathbf{x}}^{k}\right\| \leq \Lambda$.

To complete the rest of the proof, let

$$
\begin{align*}
& H(x, y)=\mathcal{L}\left(x, y, \lambda_{\mathbf{x}}^{k}, \lambda_{\mathbf{y}}^{k} ; \rho_{k}\right), \quad H^{*}=\min _{x} \max _{y} \mathcal{L}\left(x, y, \lambda_{\mathbf{x}}^{k}, \lambda_{\mathbf{y}}^{k} ; \rho_{k}\right)  \tag{70}\\
& H_{\text {low }}=\min _{(x, y) \in \mathcal{X} \times \mathcal{Y}} \mathcal{L}\left(x, y, \lambda_{\mathbf{x}}^{k}, \lambda_{\mathbf{y}}^{k} ; \rho_{k}\right) \tag{71}
\end{align*}
$$

In view of these, (59), (67), (68), (69), we obtain that

$$
\begin{aligned}
& \max _{y} H\left(x_{\mathrm{init}}^{k}, y\right) \stackrel{(59)}{\leq} L_{F} D_{\mathbf{y}}+F_{\mathrm{hi}}+\Lambda+\frac{1}{2}\left(\tau^{-1}+\left\|\lambda_{\mathbf{y}}^{0}\right\|^{2}\right)+\frac{F_{\mathrm{hi}}-f_{\mathrm{low}}^{*}+D_{\mathbf{y}} \epsilon_{0}}{1-\tau}, \\
& f_{\text {low }}^{*}-\frac{\Lambda^{2}}{2 \rho_{k}} \stackrel{\sqrt{67})}{\leq} H^{*} \stackrel{(68)}{\leq} F_{\mathrm{hi}}+\frac{1}{2}\left\|\lambda_{\mathbf{y}}^{0}\right\|^{2}+\frac{F_{\mathrm{hi}}-f_{\text {low }}^{*}+D_{\mathbf{y}} \epsilon_{0}}{1-\tau} \\
& H_{\text {low }} \\
& \stackrel{(69)}{\geq} F_{\text {low }}-\frac{\Lambda^{2}}{2 \rho_{k}}-\left\|\lambda_{\mathbf{y}}^{0}\right\|^{2}-\frac{2\left(F_{\mathrm{hi}}-f_{\mathrm{low}}^{*}+D_{\mathbf{y}} \epsilon_{0}\right)}{1-\tau}-\rho_{k} d_{\mathrm{hi}}^{2}
\end{aligned}
$$

Using these and Theorem 2 (see Appendix A) with $x^{0}=x_{\text {init }}^{k}, D_{p}=D_{\mathbf{x}}, D_{q}=D_{\mathbf{y}}, \epsilon=\epsilon_{k}$, $\epsilon_{0}=\epsilon_{k} /\left(2 \sqrt{\rho_{k}}\right), L_{\nabla h}=L_{k}, \alpha=\alpha_{k}, \delta=\delta_{k}$, and $H, H^{*}, H_{\text {low }}$ given in (70) and (71), we can conclude that Algorithm 3 performs at most $N_{k}$ evaluations of $\nabla f, \nabla c, \nabla d$ and proximal operator of $p$ and $q$ for finding an $\epsilon_{k}$-stationary point of problem (9) satisfying (66).
Lemma 7. Suppose that Assumptions 1, 园 and 3 hold. Let $f_{\text {low }}^{*}, D_{\mathbf{y}}, F_{\text {hi }}$ and $\widehat{L}$ be defined in (18), (19), (20) and (23), $L_{F}, L_{c}, \delta_{c}, \theta_{f}$ and $\theta_{a}$ be given in Assumptions 1 and (3, and $\epsilon_{0}, \tau$, $\rho_{k}, \Lambda$ and $\lambda_{\mathbf{y}}^{0}$ be given in Algorithm 1. Suppose that $\left(x^{k+1}, \lambda_{\mathbf{x}}^{k+1}\right)$ is generated by Algorithm 1 for some $0 \leq k \in \mathbb{K}-1$ with

$$
\begin{align*}
\rho_{k} \geq \max \{ & \theta_{a}^{-1} \Lambda, \theta_{f}^{-2}\left\{2 L_{F} D_{\mathbf{y}}+2 F_{\mathrm{hi}}-2 f_{\mathrm{low}}^{*}+2 \Lambda+\tau^{-1}+\left\|\lambda_{\mathbf{y}}^{0}\right\|^{2}+\frac{2\left(F_{\mathrm{hi}}-f_{\mathrm{low}}^{*}+D_{\mathbf{y}} \epsilon_{0}\right)}{1-\tau}\right. \\
& \left.\left.+\frac{\epsilon_{0} D_{\mathbf{y}}}{2}+L_{c}^{-2}+4 D_{\mathbf{y}}^{2} \widehat{L}+\Lambda^{2}\right\}, \frac{4\left\|\lambda_{\mathbf{y}}^{0}\right\|^{2}}{\delta_{d}^{2} \tau}+\frac{8\left(F_{\mathrm{hi}}-f_{\mathrm{low}}^{*}+D_{\mathbf{y}} \epsilon_{0}\right)}{\delta_{d}^{2} \tau(1-\tau)}\right\} \tag{72}
\end{align*}
$$

Let

$$
\begin{equation*}
\tilde{\lambda}_{\mathbf{x}}^{k+1}=\left[\lambda_{\mathbf{x}}^{k}+\rho_{k} c\left(x^{k+1}\right)\right]_{+} \tag{73}
\end{equation*}
$$

Then we have

$$
\begin{align*}
& \left\|\left[c\left(x^{k+1}\right)\right]_{+}\right\| \leq \rho_{k}^{-1} \delta_{c}^{-1}\left(L_{F}+2 L_{d} \delta_{d}^{-1}\left(\epsilon_{0}+L_{F}\right) D_{\mathbf{y}}+\epsilon_{0}\right)  \tag{74}\\
& \left|\left\langle\tilde{\lambda}_{\mathbf{x}}^{k+1}, c\left(x^{k+1}\right)\right\rangle\right| \leq \rho_{k}^{-1} \delta_{c}^{-1}\left(L_{F}+2 L_{d} \delta_{d}^{-1}\left(\epsilon_{0}+L_{F}\right) D_{\mathbf{y}}+\epsilon_{0}\right) \max \left\{\delta_{c}^{-1}\left(L_{F}+2 L_{d} \delta_{d}^{-1}\left(\epsilon_{0}+L_{F}\right) D_{\mathbf{y}}+\epsilon_{0}\right), \Lambda\right\} \tag{75}
\end{align*}
$$

Proof. One can observe from (3), (18), (36) and (37) that

$$
\begin{aligned}
\max _{y} \mathcal{L}\left(x^{k+1}, y, \lambda_{\mathbf{x}}^{k}, \lambda_{\mathbf{y}}^{k} ; \rho_{k}\right) & =\max _{y} \mathcal{L}_{\mathbf{y}}\left(x^{k+1}, y, \lambda_{\mathbf{y}}^{k} ; \rho_{k}\right)+\frac{1}{2 \rho_{k}}\left(\left\|\left[\lambda_{\mathbf{x}}^{k}+\rho_{k} c\left(x^{k+1}\right)\right]_{+}\right\|^{2}-\left\|\lambda_{\mathbf{x}}^{k}\right\|^{2}\right) \\
& \stackrel{\text { (37] }}{\geq} f^{*}\left(x^{k+1}\right)+\frac{1}{2 \rho_{k}}\left(\left\|\left[\lambda_{\mathbf{x}}^{k}+\rho_{k} c\left(x^{k+1}\right)\right]_{+}\right\|^{2}-\left\|\lambda_{\mathbf{x}}^{k}\right\|^{2}\right) \\
& \stackrel{\text { II8) }}{\geq} f_{\text {low }}^{*}+\frac{1}{2 \rho_{k}}\left(\left\|\left[\lambda_{\mathbf{x}}^{k}+\rho_{k} c\left(x^{k+1}\right)\right]_{+}\right\|^{2}-\left\|\lambda_{\mathbf{x}}^{k}\right\|^{2}\right) .
\end{aligned}
$$

By this inequality, (66) and $\left\|\lambda_{\mathrm{x}}^{k}\right\| \leq \Lambda$, one has

$$
\begin{aligned}
& \left\|\left[\lambda_{\mathbf{x}}^{k}+\rho_{k} c\left(x^{k+1}\right)\right]_{+}\right\|^{2} \leq 2 \rho_{k} \max _{y} \mathcal{L}\left(x^{k+1}, y, \lambda_{\mathbf{x}}^{k}, \lambda_{\mathbf{y}}^{k} ; \rho_{k}\right)-2 \rho_{k} f_{\text {low }}^{*}+\left\|\lambda_{\mathbf{x}}^{k}\right\|^{2} \\
& \leq 2 \rho_{k} \max _{y} \mathcal{L}\left(x^{k+1}, y, \lambda_{\mathbf{x}}^{k}, \lambda_{\mathbf{y}}^{k} ; \rho_{k}\right)-2 \rho_{k} f_{\text {low }}^{*}+\Lambda^{2} \\
& \stackrel{(66)}{\leq} 2 \rho_{k} L_{F} D_{\mathbf{y}}+2 \rho_{k} F_{\mathrm{hi}}+2 \rho_{k} \Lambda+\rho_{k}\left(\tau^{-1}+\left\|\lambda_{\mathbf{y}}^{0}\right\|^{2}\right)+\frac{2 \rho_{k}\left(F_{\mathrm{hi}}-f_{\text {low }}^{*}+D_{\mathbf{y}} \epsilon_{0}\right)}{1-\tau}+\frac{\rho_{k} \epsilon_{k} D_{\mathbf{y}}}{2} \\
& \quad+L_{k}^{-1} \epsilon_{k}^{2}+4 D_{\mathbf{y}}^{2} L_{k}-2 \rho_{k} f_{\text {low }}^{*}+\Lambda^{2}
\end{aligned}
$$

This together with $\rho_{k}^{2}\left\|\left[c\left(x^{k+1}\right)\right]_{+}\right\|^{2} \leq\left\|\left[\lambda_{\mathbf{x}}^{k}+\rho_{k} c\left(x^{k+1}\right)\right]_{+}\right\|^{2}$ implies that

$$
\begin{align*}
\left\|\left[c\left(x^{k+1}\right)\right]_{+}\right\|^{2} \leq & \rho_{k}^{-1}\left(2 L_{F} D_{\mathbf{y}}+2 F_{\mathrm{hi}}-2 f_{\text {low }}^{*}+2 \Lambda+\tau^{-1}+\left\|\lambda_{\mathbf{y}}^{0}\right\|^{2}+\frac{2\left(F_{\mathrm{hi}}-f_{\mathrm{low}}^{*}+D_{\mathbf{y}} \epsilon_{0}\right)}{1-\tau}+\frac{\epsilon_{k} D_{\mathbf{y}}}{2}\right) \\
& +\rho_{k}^{-2}\left(L_{k}^{-1} \epsilon_{k}^{2}+4 D_{\mathbf{y}}^{2} L_{k}+\Lambda^{2}\right) . \tag{76}
\end{align*}
$$

In addition, we observe from (10), (23), (41), $\rho_{k} \geq 1$ and $\left\|\lambda_{\mathrm{x}}^{k}\right\| \leq \Lambda$ that for all $0 \leq k \leq K$,

$$
\begin{align*}
& \rho_{k} L_{c}^{2} \leq L_{k}=L_{\nabla f}+\rho_{k} L_{c}^{2}+\rho_{k} c_{\mathrm{hi}} L_{\nabla c}+\left\|\lambda_{\mathbf{x}}^{k}\right\| L_{\nabla c}+\rho_{k} L_{d}^{2}+\rho_{k} d_{\mathrm{hi}} L_{\nabla d}+\left\|\lambda_{\mathbf{y}}^{k}\right\| L_{\nabla d} \\
& \leq L_{\nabla f}+\rho_{k} L_{c}^{2}+\rho_{k} c_{\mathrm{hi}} L_{\nabla c}+\Lambda L_{\nabla c}+\rho_{k} L_{d}^{2}+\rho_{k} d_{\mathrm{hi}} L_{\nabla d} \\
& \quad+L_{\nabla d} \sqrt{\rho_{k}\left(\left\|\lambda_{\mathbf{y}}^{0}\right\|^{2}+\frac{2\left(F_{\mathrm{hi}}-f_{\mathrm{low}}^{*}+D_{y} \epsilon_{0}\right)}{1-\tau}\right)} \leq \rho_{k} \widehat{L} \tag{77}
\end{align*}
$$

Using this relation, (72), (76), $\rho_{k} \geq 1$ and $\epsilon_{k} \leq \epsilon_{0}$, we have

$$
\begin{aligned}
\left\|\left[c\left(x^{k+1}\right)\right]_{+}\right\|^{2} \leq & \rho_{k}^{-1}\left(2 L_{F} D_{\mathbf{y}}+2 F_{\mathrm{hi}}-f_{\text {low }}^{*}+2 \Lambda+\tau^{-1}+\left\|\lambda_{\mathbf{y}}^{0}\right\|^{2}+\frac{2\left(F_{\mathrm{hi}}-f_{\text {low }}^{*}+D_{\mathbf{y}} \epsilon_{0}\right)}{1-\tau}+\frac{\epsilon_{k} D_{\mathbf{y}}}{2}\right) \\
& +\rho_{k}^{-2}\left(\left(\rho_{k} L_{c}^{2}\right)^{-1} \epsilon_{k}^{2}+4 \rho_{k} D_{\mathbf{y}}^{2} \widehat{L}+\Lambda^{2}\right) \\
\leq & \rho_{k}^{-1}\left(2 L_{F} D_{\mathbf{y}}+2 F_{\mathrm{hi}}-f_{\text {low }}^{*}+2 \Lambda+\tau^{-1}+\left\|\lambda_{\mathbf{y}}^{0}\right\|^{2}+\frac{2\left(F_{\mathrm{hi}}-f_{\text {low }}^{*}+D_{\mathbf{y}} \epsilon_{0}\right)}{1-\tau}+\frac{\epsilon_{0} D_{\mathbf{y}}}{2}\right) \\
& +\rho_{k}^{-1}\left(L_{c}^{-2}+4 D_{\mathbf{y}}^{2} \widehat{L}+\Lambda^{2}\right) \stackrel{[72]}{\leq} \theta_{f}^{2},
\end{aligned}
$$

which together with (111) implies that $x^{k+1} \in \mathcal{F}\left(\theta_{f}\right)$.
It follows from $x^{k+1} \in \mathcal{F}\left(\theta_{f}\right)$ and Assumption [3(i) that there exists some $v_{x}$ such that $\left\|v_{x}\right\|=1$ and $v_{x}^{T} \nabla c_{i}\left(x^{k+1}\right) \leq-\delta_{c}$ for all $i \in \mathcal{A}\left(x^{k+1} ; \theta_{a}\right)$, where $\mathcal{A}\left(x^{k+1} ; \theta_{a}\right)$ is defined in (111). Let $\overline{\mathcal{A}}\left(x^{k+1} ; \theta_{a}\right)=\{1,2, \ldots, \tilde{n}\} \backslash \mathcal{A}\left(x^{k+1} ; \theta_{a}\right)$. Notice from (11) that $c_{i}\left(x^{k+1}\right)<-\theta_{a}$ for all $i \in \overline{\mathcal{A}}\left(x^{k+1} ; \theta_{a}\right)$. In addition, observe from (72) that $\rho_{k} \geq \theta_{a}^{-1} \Lambda$. Using these and $\left\|\lambda_{\mathbf{x}}^{k}\right\| \leq \Lambda$, we obtain that $\left(\lambda_{\mathbf{x}}^{k}+\rho_{k} c\left(x^{k+1}\right)\right)_{i} \leq \Lambda-\rho_{k} \theta_{a} \leq 0$ for all $i \in \overline{\mathcal{A}}\left(x^{k+1} ; \theta_{a}\right)$. By this and the fact that $v_{x}^{T} \nabla c_{i}\left(x^{k+1}\right) \leq-\delta_{c}$ for all $i \in \mathcal{A}\left(x^{k+1} ; \theta_{a}\right)$, one has

$$
\begin{align*}
& v_{x}^{T} \nabla c\left(x^{k+1}\right) \tilde{\lambda}_{\mathbf{x}}^{k+1} \stackrel{(733)}{=} v_{x}^{T} \nabla c\left(x^{k+1}\right)\left[\lambda_{\mathbf{x}}^{k}+\rho_{k} c\left(x^{k+1}\right)\right]_{+}=\sum_{i=1}^{\tilde{n}} v_{x}^{T} \nabla c_{i}\left(x^{k+1}\right)\left(\left[\lambda_{\mathbf{x}}^{k}+\rho_{k} c\left(x^{k+1}\right)\right]_{+}\right)_{i} \\
& =\sum_{i \in \mathcal{A}\left(x^{k+1} ; \theta_{a}\right)} v_{x}^{T} \nabla c_{i}\left(x^{k+1}\right)\left(\left[\lambda_{\mathbf{x}}^{k}+\rho_{k} c\left(x^{k+1}\right)\right]_{+}\right)_{i}+\sum_{i \in \overline{\mathcal{A}}\left(x^{k+1 ;} ; \theta_{a}\right)} v_{x}^{T} \nabla c_{i}\left(x^{k+1}\right)\left(\left[\lambda_{\mathbf{x}}^{k}+\rho_{k} c\left(x^{k+1}\right)\right]_{+}\right)_{i} \\
& \leq-\delta_{c} \sum_{i \in \mathcal{A}\left(x^{k+1} ; \theta_{a}\right)}\left(\left[\lambda_{\mathbf{x}}^{k}+\rho_{k} c\left(x^{k+1}\right)\right]_{+}\right)_{i}=-\delta_{c} \sum_{i=1}^{\tilde{n}}\left(\left[\lambda_{\mathbf{x}}^{k}+\rho_{k} c\left(x^{k+1}\right)\right]_{+}\right)_{i} \stackrel{(73)}{=}-\delta_{c}\left\|\tilde{\lambda}_{\mathbf{x}}^{k+1}\right\|_{1} . \tag{78}
\end{align*}
$$

Since ( $x^{k+1}, y^{k+1}$ ) is an $\epsilon_{k}$-stationary point of (9), it follows from (3) and (56) that there exists some $s \in \partial_{x} F\left(x^{k+1}, y^{k+1}\right)$ such that

$$
\left\|s+\nabla c\left(x^{k+1}\right)\left[\lambda_{\mathbf{x}}^{k}+\rho_{k} c\left(x^{k+1}\right)\right]_{+}-\nabla_{x} d\left(x^{k+1}, y^{k+1}\right)\left[\lambda_{\mathbf{y}}^{k}+\rho_{k} d\left(x^{k+1}, y^{k+1}\right)\right]_{+}\right\| \leq \epsilon_{k},
$$

which along with（731）and $\lambda_{\mathbf{y}}^{k+1}=\left[\lambda_{\mathbf{y}}^{k}+\rho_{x} d\left(x^{k+1}, y^{k+1}\right)\right]_{+}$implies that

$$
\left\|s+\nabla c\left(x^{k+1}\right) \tilde{\lambda}_{\mathbf{x}}^{k+1}-\nabla_{x} d\left(x^{k+1}, y^{k+1}\right) \lambda_{\mathbf{y}}^{k+1}\right\| \leq \epsilon_{k} .
$$

By this，（78）and $\left\|v_{x}\right\|=1$ ，one has

$$
\begin{aligned}
\epsilon_{k} & \geq\left\|s+\nabla c\left(x^{k+1}\right) \tilde{\lambda}_{\mathbf{x}}^{k+1}-\nabla_{x} d\left(x^{k+1}, y^{k+1}\right) \lambda_{\mathbf{y}}^{k+1}\right\| \cdot\left\|v_{x}\right\| \\
& \geq\left\langle s+\nabla c\left(x^{k+1}\right) \tilde{\lambda}_{\mathbf{x}}^{k+1}-\nabla_{x} d\left(x^{k+1}, y^{k+1}\right) \lambda_{\mathbf{y}}^{k+1},-v_{x}\right\rangle \\
& =-\left\langle s-\nabla_{x} d\left(x^{k+1}, y^{k+1}\right) \lambda_{\mathbf{y}}^{k+1}, v_{x}\right\rangle-v_{x}^{T} \nabla c\left(x^{k+1}\right) \tilde{\lambda}_{\mathbf{x}}^{k+1} \\
& \stackrel{\text { (788) }}{\geq}-\left(\|s\|+\left\|\nabla_{x} d\left(x^{k+1}, y^{k+1}\right)\right\|\left\|\lambda_{\mathbf{y}}^{k+1}\right\|\right)\left\|v_{x}\right\|+\delta_{c}\left\|\tilde{\lambda}_{\mathbf{x}}^{k+1}\right\|_{1} . \\
& \geq-L_{F}-L_{d}\left\|\lambda_{\mathbf{y}}^{k+1}\right\|+\delta_{c}\left\|\tilde{\lambda}_{\mathbf{x}}^{k+1}\right\|_{1},
\end{aligned}
$$

where the last inequality is due to $\left\|v_{x}\right\|=1$ and Assumptions（i）and（iii）．Notice from（72） that（44）holds．It then follows from（45））that $\left\|\lambda_{\mathbf{y}}^{k+1}\right\| \leq 2 \delta_{d}^{-1}\left(\epsilon_{0}+L_{F}\right) D_{\mathbf{y}}$ ，which together with the above inequality and $\epsilon_{k} \leq \epsilon_{0}$ yields

$$
\begin{equation*}
\left\|\tilde{\lambda}_{\mathbf{x}}^{k+1}\right\| \leq\left\|\tilde{\lambda}_{\mathbf{x}}^{k+1}\right\|_{1} \leq \delta_{c}^{-1}\left(L_{F}+L_{d}\left\|\lambda_{\mathbf{y}}^{k+1}\right\|+\epsilon_{k}\right) \leq \delta_{c}^{-1}\left(L_{F}+2 L_{d} \delta_{d}^{-1}\left(\epsilon_{0}+L_{F}\right) D_{\mathbf{y}}+\epsilon_{0}\right) . \tag{79}
\end{equation*}
$$

By this and（73），one can observe that
$\left\|\left[c\left(x^{k+1}\right)\right]_{+}\right\| \leq \rho_{k}^{-1}\left\|\left[\lambda_{\mathbf{x}}^{k}+\rho_{k} c\left(x^{k+1}\right)\right]_{+}\right\|=\rho_{k}^{-1}\left\|\tilde{\lambda}_{\mathbf{x}}^{k+1}\right\| \leq \rho_{k}^{-1} \delta_{c}^{-1}\left(L_{F}+2 L_{d} \delta_{d}^{-1}\left(\epsilon_{0}+L_{F}\right) D_{\mathbf{y}}+\epsilon_{0}\right)$.
Hence，（744）holds as desired．
We next show that（75）holds．Indeed，by $\tilde{\lambda}_{\mathrm{x}}^{k+1} \geq 0$ ，（74）and（79），one has

$$
\begin{align*}
\left\langle\tilde{\lambda}_{\mathbf{x}}^{k+1}, c\left(x^{k+1}\right)\right\rangle & \leq\left\langle\tilde{\lambda}_{\mathrm{x}}^{k+1},\left[c\left(x^{k+1}\right)\right]_{+}\right\rangle \leq\left\|\tilde{\lambda}_{\mathrm{x}}^{k+1}\right\|\left\|\left[c\left(x^{k+1}\right)\right]_{+}\right\| \\
& \stackrel{(774)}{\leq} \rho_{k}^{-1} \delta_{c}^{-2}\left(L_{F}+2 L_{d} \delta_{d}^{-1}\left(\epsilon_{0}+L_{F}\right) D_{\mathbf{y}}+\epsilon_{0}\right)^{2} . \tag{80}
\end{align*}
$$

Using a similar argument as for the proof of（47），we have

$$
-\left\langle\tilde{\lambda}_{\mathbf{x}}^{k+1}, \rho_{k}^{-1} \lambda_{\mathbf{x}}^{k}\right\rangle \leq\left\langle\tilde{\lambda}_{\mathbf{x}}^{k+1}, c\left(x^{k+1}\right)\right\rangle,
$$

which along with $\left\|\lambda_{\mathbf{x}}^{k}\right\| \leq \Lambda$ and（79）yields

$$
\left\langle\tilde{\lambda}_{\mathbf{x}}^{k+1}, c\left(x^{k+1}\right)\right\rangle \geq-\rho_{k}^{-1}\left\|\tilde{\lambda}_{\mathbf{x}}^{k+1}\right\|\left\|\lambda_{\mathbf{x}}^{k}\right\| \geq-\rho_{k}^{-1} \delta_{c}^{-1}\left(L_{F}+2 L_{d} \delta_{d}^{-1}\left(\epsilon_{0}+L_{F}\right) D_{\mathbf{y}}+\epsilon_{0}\right) \Lambda .
$$

The relation（75）then follows from this and（80）．
We are now ready to prove Theorem 1 ，
Proof of Theorem 11．（i）Observe from the definition of $K$ in（22）and $\epsilon_{k}=\epsilon_{0} \tau^{k}$ that $K$ is the smallest nonnegative integer such that $\epsilon_{K} \leq \varepsilon$ ．Hence，Algorithm 1 terminates and outputs $\left(x^{K+1}, y^{K+1}\right)$ after $K+1$ outer iterations．It follows from these and $\rho_{k}=\epsilon_{k}^{-1}$ that $\epsilon_{K} \leq \varepsilon$ and $\rho_{K} \geq \varepsilon^{-1}$ ．By this and（28），one can see that（50）and（72）holds for $k=K$ ．It then follows from Lemmas 4 and 7 that（29）－（34）hold．
（ii）Let $K$ and $N$ be given in（22）and（35）．Recall from Lemma 6 that the number of evaluations of $\nabla f, \nabla c, \nabla d$ ，proximal operator of $p$ and $q$ performed by Algorithm 3at iteration $k$ of Algorithm is at most $N_{k}$ ，where $N_{k}$ is given in（65）．By this and statement（i）of this theorem，one can observe that the total number of evaluations of $\nabla f, \nabla c, \nabla d$ ，proximal operator of $p$ and $q$ performed in Algorithm $⿴ 囗 十 ⺝$ is no more than $\sum_{k=0}^{K} N_{k}$ ，respectively．As a result，to prove statement（ii）of this theorem，it suffices to show that $\sum_{k=0}^{K} N_{k} \leq N$ ．Recall from（77）
and Algorithm 1 that $\rho_{k} L_{c}^{2} \leq L_{k} \leq \rho_{k} \widehat{L}$ and $\rho_{k} \geq 1 \geq \epsilon_{k}$. Using these, (24), (25), (26), (61), (62), (63) and (64), we obtain that

$$
\begin{align*}
& 1 \geq \alpha_{k} \geq \min \left\{1, \sqrt{4 \epsilon_{k} /\left(\rho_{k} D_{\mathbf{y}} \widehat{L}\right)}\right\} \geq \epsilon_{k}^{1 / 2} \rho_{k}^{-1 / 2} \hat{\alpha},  \tag{81}\\
& \delta_{k} \leq\left(2+\epsilon_{k}^{-1 / 2} \rho_{k}^{1 / 2} \hat{\alpha}^{-1}\right) \rho_{k} \widehat{L} D_{\mathbf{x}}^{2}+\max \left\{1 / D_{\mathbf{y}}, \rho_{k} \widehat{L} / 4\right\} D_{\mathbf{y}}^{2} \leq \epsilon_{k}^{-1 / 2} \rho_{k}^{3 / 2} \hat{\delta},  \tag{82}\\
& M_{k} \leq \frac{16 \max \left\{1 /\left(2 \rho_{k} L_{c}^{2}\right), 4 /\left(\epsilon_{k}^{1 / 2} \rho_{k}^{-1 / 2} \hat{\alpha} \rho_{k} L_{c}^{2}\right)\right\} \rho_{k}}{\left[\left(3 \rho_{k} \widehat{L}+1 /\left(2 D_{\mathbf{y}}\right)\right)^{2} / \min \left\{\rho_{k} L_{c}^{2}, \epsilon_{k} /\left(2 D_{\mathbf{y}}\right)\right\}+3 \rho_{k} \widehat{L}+1 /\left(2 D_{\mathbf{y}}\right)\right]^{-2} \epsilon_{k}^{2}} \times\left(\epsilon_{k}^{-1 / 2} \rho_{k}^{3 / 2} \hat{\delta}\right. \\
&+2 \epsilon_{k}^{-1 / 2} \rho_{k}^{1 / 2} \hat{\alpha}^{-1}\left(F_{\mathrm{hi}}-F_{\text {low }}+\frac{\Lambda^{2}}{2}+\frac{3}{2}\left\|\lambda_{\mathbf{y}}^{0}\right\|^{2}+\frac{3\left(F_{\mathrm{hi}}-f_{\mathrm{low}}^{*}+D_{\mathbf{y}} \epsilon_{0}\right)}{1-\tau}+\rho_{k} d_{\mathrm{hi}}^{2}\right. \\
&\left.\left.+\frac{D_{\mathbf{y}}}{4}+\rho_{k} \widehat{L} D_{\mathbf{x}}^{2}\right)\right)  \tag{83}\\
& \leq \frac{16 \epsilon_{k}^{-1 / 2} \rho_{k}^{-1 / 2} \max \left\{1 /\left(2 L_{c}^{2}\right), 4 /\left(\hat{\alpha} L_{c}^{2}\right)\right\} \rho_{k}}{\epsilon_{k}^{2} \rho_{k}^{-4}\left[\left(3 \widehat{L}+1 /\left(2 D_{\mathbf{y}}\right)\right)^{2} / \min \left\{L_{c}^{2}, 1 /\left(2 D_{\mathbf{y}}\right)\right\}+3 \widehat{L}+1 /\left(2 D_{\mathbf{y}}\right)\right]^{-2} \epsilon_{k}^{2}} \times\left(\epsilon_{k}^{-1 / 2} \rho_{k}^{3 / 2}\right)\left(\hat{\delta}+2 \hat{\alpha}^{-1}\right. \\
& \times\left.\left(F_{\mathrm{hi}}-F_{\mathrm{low}}+\frac{\Lambda^{2}}{2}+\frac{3}{2}\left\|\lambda_{\mathbf{y}}^{0}\right\|^{2}+\frac{3\left(F_{\mathrm{hi}}-f_{\text {low }}^{*}+D_{\mathbf{y}} \epsilon_{0}\right)}{1-\tau}+d_{\mathrm{hi}}^{2}+\frac{D_{\mathbf{y}}}{4}+\widehat{L} D_{\mathbf{x}}^{2}\right)\right) \leq \epsilon_{k}^{-5} \rho_{k}^{6} \widehat{M}, \\
& T_{k} \leq {\left[16\left(L_{F} D_{\mathbf{y}}+F_{\mathrm{hi}}-f_{\text {low }}^{*}+\Lambda+\frac{1}{2}\left(\tau^{-1}+\left\|\lambda_{\mathbf{y}}^{0}\right\|^{2}\right)+\frac{F_{\mathrm{hi}}-f_{\mathrm{low}}^{*}+D_{\mathbf{y}} \epsilon_{0}}{1-\tau}+\frac{\Lambda^{2}}{2}+\frac{D_{\mathbf{y}}}{4}\right) \epsilon_{k}^{-2} \rho_{k} \widehat{L}\right.} \\
&\left.+8\left(1+4 D_{\mathbf{y}}^{2} \rho_{k}^{2} \widehat{L}^{2} \epsilon_{k}^{-2}\right) \rho_{k}^{-1}-1\right] \quad \leq \epsilon_{k}^{-2} \rho_{k} \widehat{T},
\end{align*}
$$

where (83) follows from (24), (25), (26), (81), (82), $\rho_{k} L_{c}^{2} \leq L_{k} \leq \rho_{k} \widehat{L}$, and $\rho_{k} \geq 1 \geq \epsilon_{k}$. By the above inequalities, (65), (77), $\widehat{T} \geq 1$ and $\rho_{k} \geq 1 \geq \epsilon_{k}$, one has

$$
\begin{align*}
& \sum_{k=0}^{K} N_{k} \leq \sum_{k=0}^{K}\left(\left[96 \sqrt{2}\left(1+\left(24 \rho_{k} \widehat{L}+4 / D_{\mathbf{y}}\right) /\left(\rho_{k} L_{c}^{2}\right)\right)\right]+2\right) \max \left\{2, \sqrt{D_{\mathbf{y}} \rho_{k} \widehat{L} \epsilon_{k}^{-1}}\right\} \\
& \quad \times\left(\left(\epsilon_{k}^{-2} \rho_{k} \widehat{T}+1\right)\left(\log \left(\epsilon_{k}^{-5} \rho_{k}^{6} \widehat{M}\right)\right)_{+}+\epsilon_{k}^{-2} \rho_{k} \widehat{T}+1+2 \epsilon_{k}^{-2} \rho_{k} \widehat{T} \log \left(\epsilon_{k}^{-2} \rho_{k} \widehat{T}+1\right)\right) \\
& \leq \sum_{k=0}^{K}\left(\left[96 \sqrt{2}\left(1+\left(24 \widehat{L}+4 / D_{\mathbf{y}}\right) / L_{c}^{2}\right)\right]+2\right) \max \left\{2, \sqrt{D_{\mathbf{y}} \widehat{L}}\right\} \epsilon_{k}^{-1 / 2} \rho_{k}^{1 / 2} \\
& \quad \times \epsilon_{k}^{-2} \rho_{k}\left((\widehat{T}+1)\left(\log \left(\epsilon_{k}^{-5} \rho_{k}^{6} \widehat{M}\right)\right)_{+}+\widehat{T}+1+2 \widehat{T} \log \left(\epsilon_{k}^{-2} \rho_{k} \widehat{T}+1\right)\right) \\
& \leq \\
& \sum_{k=0}^{K}\left(\left[96 \sqrt{2}\left(1+\left(24 \widehat{L}+4 / D_{\mathbf{y}}\right) / L_{c}^{2}\right)\right]+2\right) \max \left\{2, \sqrt{D_{\mathbf{y}} \widehat{L}}\right\} \\
& \quad \times \epsilon_{k}^{-5 / 2} \rho_{k}^{3 / 2} \widehat{T}\left(2\left(\log \left(\epsilon_{k}^{-5} \rho_{k}^{6} \widehat{M}\right)\right)_{+}+2+2 \log \left(2 \epsilon_{k}^{-2} \rho_{k} \widehat{T}\right)\right) \\
& \leq  \tag{84}\\
& \\
& \quad \sum_{k=0}^{K}\left(\left[96 \sqrt{2}\left(1+\left(24 \widehat{L}+4 / D_{\mathbf{y}}\right) / L_{c}^{2}\right)\right]+2\right) \max \left\{2, \sqrt{D_{\mathbf{y}} \widehat{L}}\right\} \widehat{T} \\
& \quad \times \epsilon_{k}^{-5 / 2} \rho_{k}^{3 / 2}\left(14 \log \rho_{k}-14 \log \epsilon_{k}+2(\log \widehat{M})_{+}+2+2 \log (2 \widehat{T})\right)
\end{align*}
$$

By the definition of $K$ in (22), one has $\tau^{K} \geq \tau \varepsilon / \epsilon_{0}$. Also, notice from Algorithm 1 that
$\rho_{k}=\epsilon_{k}^{-1}=\left(\epsilon_{0} \tau^{k}\right)^{-1}$. It then follows from these, (35) and (84) that

$$
\begin{aligned}
& \sum_{k=0}^{K} N_{k} \leq \sum_{k=0}^{K}\left(\left[96 \sqrt{2}\left(1+\left(24 \widehat{L}+4 / D_{y}\right) / L_{c}^{2}\right)\right\rceil+2\right) \max \left\{2, \sqrt{D_{y} \widehat{L}}\right\} \widehat{T} \\
& \times \epsilon_{k}^{-4}\left(28 \log \left(1 / \epsilon_{k}\right)+2(\log \widehat{M})_{+}+2+2 \log (2 \widehat{T})\right) \\
&=\left(\left[96 \sqrt{2}\left(1+\left(24 \widehat{L}+4 / D_{y}\right) / L_{c}^{2}\right)\right]+2\right) \max \left\{2, \sqrt{D_{y} \widehat{L}}\right\} \widehat{T} \\
& \times \sum_{k=0}^{K} \epsilon_{0}^{-4} \tau^{-4 k}\left(28 k \log (1 / \tau)+28 \log \left(1 / \epsilon_{0}\right)+2(\log \widehat{M})_{+}+2+2 \log (2 \widehat{T})\right) \\
& \leq\left(\left[96 \sqrt{2}\left(1+\left(24 \widehat{L}+4 / D_{y}\right) / L_{c}^{2}\right)\right]+2\right) \max \left\{2, \sqrt{D_{y} \widehat{L}}\right\} \widehat{T} \\
& \quad \times \sum_{k=0}^{K} \epsilon_{0}^{-4} \tau^{-4 k}\left(28 K \log (1 / \tau)+28 \log \left(1 / \epsilon_{0}\right)+2(\log \widehat{M})_{+}+2+2 \log (2 \widehat{T})\right) \\
& \leq\left(\left[96 \sqrt{2}\left(1+\left(24 \widehat{L}+4 / D_{y}\right) / L_{c}^{2}\right)\right]+2\right) \max \left\{2, \sqrt{D_{y} \widehat{L}}\right\} \widehat{T} \epsilon_{0}^{-4} \\
& \quad \times \tau^{-4 K}\left(1-\tau^{4}\right)^{-1}\left(28 K \log (1 / \tau)+28 \log \left(1 / \epsilon_{0}\right)+2(\log \widehat{M})_{+}+2+2 \log (2 \widehat{T})\right) \\
& \leq\left(\left[96 \sqrt{2}\left(1+\left(24 \widehat{L}+4 / D_{y}\right) / L_{c}^{2}\right)\right]+2\right) \max \left\{2, \sqrt{D_{y} \widehat{L}}\right\} \widehat{T} \epsilon_{0}^{-4}\left(1-\tau^{4}\right)^{-1} \\
& \times\left(\tau \varepsilon / \epsilon_{0}\right)^{-4}\left(28 K \log (1 / \tau)+28 \log \left(1 / \epsilon_{0}\right)+2(\log \widehat{M})_{+}+2+2 \log (2 \widehat{T})\right) \stackrel{\text { (35) }}{=} N,
\end{aligned}
$$

where the second last inequality is due to $\sum_{k=0}^{K} \tau^{-4 k} \leq \tau^{-4 K} /\left(1-\tau^{4}\right)$, and the last inequality is due to $\tau^{K} \geq \tau \varepsilon / \epsilon_{0}$. Hence, statement (ii) of this theorem holds as desired.

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## A A first-order method for nonconvex-concave minimax problem

In this part we present a first-order method proposed in [26, Algorithm 2] for finding an $\epsilon$ stationary point of the nonconvex-concave minimax problem

$$
\begin{equation*}
H^{*}=\min _{x} \max _{y}\{H(x, y):=h(x, y)+p(x)-q(y)\} \tag{85}
\end{equation*}
$$

which has at least one optimal solution and satisfies the following assumptions.
Assumption 4. (i) $p: \mathbb{R}^{n} \rightarrow \mathbb{R} \cup\{\infty\}$ and $q: \mathbb{R}^{m} \rightarrow \mathbb{R} \cup\{\infty\}$ are proper convex functions and continuous on $\operatorname{dom} p$ and $\operatorname{dom} q$, respectively, and moreover, $\operatorname{dom} p$ and $\operatorname{dom} q$ are compact.
(ii) The proximal operator associated with $p$ and $q$ can be exactly evaluated.
(iii) $h$ is $L_{\nabla h}$-smooth on $\operatorname{dom} p \times \operatorname{dom} q$, and moreover, $h(x, \cdot)$ is concave for any $x \in \operatorname{dom} p$.

For ease of presentation, we define

$$
\begin{align*}
& D_{p}=\max \{\|u-v\| \| u, v \in \operatorname{dom} p\}, \quad D_{q}=\max \{\|u-v\| \| u, v \in \operatorname{dom} q\},  \tag{86}\\
& H_{\text {low }}=\min \{H(x, y) \mid(x, y) \in \operatorname{dom} p \times \operatorname{dom} q\} . \tag{87}
\end{align*}
$$

Given an iterate $\left(x^{k}, y^{k}\right)$, the first-order method [26, Algorithm 2] finds the next iterate $\left(x^{k+1}, y^{k+1}\right)$ by applying a modified optimal first-order method [26, Algorithm 1] to the strongly-convex-strongly-concave minimax problem

$$
\begin{equation*}
\min _{x} \max _{y}\left\{h_{k}(x, y)=h(x, y)-\epsilon\left\|y-y^{0}\right\|^{2} /\left(4 D_{q}\right)+L_{\nabla h}\left\|x-x^{k}\right\|^{2}\right\} . \tag{88}
\end{equation*}
$$

For ease reference, we next present the modified optimal first-order method [26, Algorithm 1] in Algorithm [2 below for solving the strongly-convex-strongly-concave minimax problem

$$
\begin{equation*}
\min _{x} \max _{y}\{\bar{h}(x, y)+p(x)-q(y)\}, \tag{89}
\end{equation*}
$$

where $\bar{h}(x, y)$ is $\sigma_{x}$-strongly-convex- $\sigma_{y}$-strongly-concave and $L_{\nabla \bar{h}}$-smooth on $\operatorname{dom} p \times \operatorname{dom} q$ for some $\sigma_{x}, \sigma_{y}>0$. In Algorithm 2, the functions $\hat{h}, a_{x}^{k}$ and $a_{y}^{k}$ are defined as follows:

$$
\begin{aligned}
& \hat{h}(x, y)=\bar{h}(x, y)-\sigma_{x}\|x\|^{2} / 2+\sigma_{y}\|y\|^{2} / 2, \\
& a_{x}^{k}(x, y)=\nabla_{x} \hat{h}(x, y)+\sigma_{x}\left(x-\sigma_{x}^{-1} z_{g}^{k}\right) / 2, \quad a_{y}^{k}(x, y)=-\nabla_{y} \hat{h}(x, y)+\sigma_{y} y+\sigma_{x}\left(y-y_{g}^{k}\right) / 8,
\end{aligned}
$$

where $y_{g}^{k}$ and $z_{g}^{k}$ are generated at iteration $k$ of Algorithm 2 below.

```
Algorithm 2 A modified optimal first-order method for problem (89)
Input: \(\tau>0, \bar{z}^{0}=z_{f}^{0} \in-\sigma_{x} \operatorname{dom} p \mathbb{4}^{4} \bar{y}^{0}=y_{f}^{0} \in \operatorname{dom} q,\left(z^{0}, y^{0}\right)=\left(\bar{z}^{0}, \bar{y}^{0}\right), \bar{\alpha}=\)
    \(\min \left\{1, \sqrt{8 \sigma_{y} / \sigma_{x}}\right\}, \quad \eta_{z}=\sigma_{x} / 2, \quad \eta_{y}=\min \left\{1 /\left(2 \sigma_{y}\right), 4 /\left(\bar{\alpha} \sigma_{x}\right)\right\}, \quad \beta_{t}=2 /(t+3), \zeta=\)
    \(\left(2 \sqrt{5}\left(1+8 L_{\nabla \bar{h}} / \sigma_{x}\right)\right)^{-1}, \gamma_{x}=\gamma_{y}=8 \sigma_{x}^{-1}\), and \(\hat{\zeta}=\min \left\{\sigma_{x}, \sigma_{y}\right\} / L_{\nabla \bar{h}}^{2}\).
    for \(k=0,1,2, \ldots\) do
        \(\left(z_{g}^{k}, y_{g}^{k}\right)=\bar{\alpha}\left(z^{k}, y^{k}\right)+(1-\bar{\alpha})\left(z_{f}^{k}, y_{f}^{k}\right)\).
        \(\left(x^{k,-1}, y^{k,-1}\right)=\left(-\sigma_{x}^{-1} z_{g}^{k}, y_{g}^{k}\right)\).
        \(x^{k, 0}=\operatorname{prox}_{\zeta \gamma_{x} p}\left(x^{k,-1}-\zeta \gamma_{x} a_{x}^{k}\left(x^{k,-1}, y^{k,-1}\right)\right)\).
        \(y^{k, 0}=\operatorname{prox}_{\zeta \gamma_{y} q}\left(y^{k,-1}-\zeta \gamma_{y} a_{y}^{k}\left(x^{k,-1}, y^{k,-1}\right)\right)\).
        \(b_{x}^{k, 0}=\frac{1}{\zeta \gamma_{x}}\left(x^{k,-1}-\zeta \gamma_{x} a_{x}^{k}\left(x^{k,-1}, y^{k,-1}\right)-x^{k, 0}\right)\).
        \(b_{y}^{k, 0}=\frac{1}{\zeta \gamma_{y}}\left(y^{k,-1}-\zeta \gamma_{y} a_{y}^{k}\left(x^{k,-1}, y^{k,-1}\right)-y^{k, 0}\right)\).
        \(t=0\).
        while
        \(\gamma_{x}\left\|a_{x}^{k}\left(x^{k, t}, y^{k, t}\right)+b_{x}^{k, t}\right\|^{2}+\gamma_{y}\left\|a_{y}^{k}\left(x^{k, t}, y^{k, t}\right)+b_{y}^{k, t}\right\|^{2}>\gamma_{x}^{-1}\left\|x^{k, t}-x^{k,-1}\right\|^{2}+\gamma_{y}^{-1}\left\|y^{k, t}-y^{k,-1}\right\|^{2}\)
            do
            \(x^{k, t+1 / 2}=x^{k, t}+\beta_{t}\left(x^{k, 0}-x^{k, t}\right)-\zeta \gamma_{x}\left(a_{x}^{k}\left(x^{k, t}, y^{k, t}\right)+b_{x}^{k, t}\right)\).
            \(y^{k, t+1 / 2}=y^{k, t}+\beta_{t}\left(y^{k, 0}-y^{k, t}\right)-\zeta \gamma_{y}\left(a_{y}^{k}\left(x^{k, t}, y^{k, t}\right)+b_{y}^{k, t}\right)\).
            \(x^{k, t+1}=\operatorname{prox}_{\zeta \gamma_{x} p}\left(x^{k, t}+\beta_{t}\left(x^{k, 0}-x^{k, t}\right)-\zeta \gamma_{x} a_{x}^{k}\left(x^{k, t+1 / 2}, y^{k, t+1 / 2}\right)\right)\).
            \(y^{k, t+1}=\operatorname{prox}_{\zeta \gamma_{y} q}\left(y^{k, t}+\beta_{t}\left(y^{k, 0}-y^{k, t}\right)-\zeta \gamma_{y} a_{y}^{k}\left(x^{k, t+1 / 2}, y^{k, t+1 / 2}\right)\right)\).
            \(b_{x}^{k, t+1}=\frac{1}{\zeta \gamma_{x}}\left(x^{k, t}+\beta_{t}\left(x^{k, 0}-x^{k, t}\right)-\zeta \gamma_{x} a_{x}^{k}\left(x^{k, t+1 / 2}, y^{k, t+1 / 2}\right)-x^{k, t+1}\right)\).
            \(b_{y}^{k, t+1}=\frac{1}{\zeta \gamma_{y}}\left(y^{k, t}+\beta_{t}\left(y^{k, 0}-y^{k, t}\right)-\zeta \gamma_{y} a_{y}^{k}\left(x^{k, t+1 / 2}, y^{k, t+1 / 2}\right)-y^{k, t+1}\right)\).
            \(t \leftarrow t+1\).
        end while
        \(\left(x_{f}^{k+1}, y_{f}^{k+1}\right)=\left(x^{k, t}, y^{k, t}\right)\).
        \(\left(z_{f}^{k+1}, w_{f}^{k+1}\right)=\left(\nabla_{x} \hat{h}\left(x_{f}^{k+1}, y_{f}^{k+1}\right)+b_{x}^{k, t},-\nabla_{y} \hat{h}\left(x_{f}^{k+1}, y_{f}^{k+1}\right)+b_{y}^{k, t}\right)\).
        \(z^{k+1}=z^{k}+\eta_{z} \sigma_{x}^{-1}\left(z_{f}^{k+1}-z^{k}\right)-\eta_{z}\left(x_{f}^{k+1}+\sigma_{x}^{-1} z_{f}^{k+1}\right)\).
        \(y^{k+1}=y^{k}+\eta_{y} \sigma_{y}\left(y_{f}^{k+1}-y^{k}\right)-\eta_{y}\left(w_{f}^{k+1}+\sigma_{y} y_{f}^{k+1}\right)\).
        \(x^{k+1}=-\sigma_{x}^{-1} z^{k+1}\).
        \(\tilde{x}^{k+1}=\operatorname{prox}_{\hat{\zeta} p}\left(x^{k+1}-\hat{\zeta} \nabla_{x} \bar{h}\left(x^{k+1}, y^{k+1}\right)\right)\).
        \(\tilde{y}^{k+1}=\operatorname{prox}_{\hat{\zeta} q}\left(y^{k+1}+\hat{\zeta} \nabla_{y} \bar{h}\left(x^{k+1}, y^{k+1}\right)\right)\).
```

        Terminate the algorithm and output ( \(\tilde{x}^{k+1}, \tilde{y}^{k+1}\) ) if
            \(\left\|\hat{\zeta}^{-1}\left(x^{k+1}-\tilde{x}^{k+1}, \tilde{y}^{k+1}-y^{k+1}\right)-\left(\nabla \bar{h}\left(x^{k+1}, y^{k+1}\right)-\nabla \bar{h}\left(\tilde{x}^{k+1}, \tilde{y}^{k+1}\right)\right)\right\| \leq \tau\).
    end for
    We are now ready to present the first-order method [26, Algorithm 2] for finding an $\epsilon$ stationary point of (85) in Algorithm 3 below.

[^4]```
Algorithm 3 A first-order method for problem (85)
    Input: \(\epsilon>0, \epsilon_{0} \in(0, \epsilon / 2],\left(\hat{x}^{0}, \hat{y}^{0}\right) \in \operatorname{dom} p \times \operatorname{dom} q,\left(x^{0}, y^{0}\right)=\left(\hat{x}^{0}, \hat{y}^{0}\right)\), and \(\epsilon_{k}=\epsilon_{0} /(k+1)\).
    for \(k=0,1,2, \ldots\) do
        Call Algorithm 2 with \(\bar{h} \leftarrow h_{k}, \tau \leftarrow \epsilon_{k}, \sigma_{x} \leftarrow L_{\nabla h}, \sigma_{y} \leftarrow \epsilon /\left(2 D_{q}\right), L_{\nabla \bar{h}} \leftarrow 3 L_{\nabla h}+\epsilon /\left(2 D_{q}\right)\),
        \(\bar{z}^{0}=z_{f}^{0} \leftarrow-\sigma_{x} x^{k}, \bar{y}^{0}=y_{f}^{0} \leftarrow y^{k}\), and denote its output by \(\left(x^{k+1}, y^{k+1}\right)\), where \(h_{k}\) is
        given in (88).
        Terminate the algorithm and output \(\left(x_{\epsilon}, y_{\epsilon}\right)=\left(x^{k+1}, y^{k+1}\right)\) if
                    \(\left\|x^{k+1}-x^{k}\right\| \leq \epsilon /\left(4 L_{\nabla h}\right)\).
```

    end for
    The following theorem presents the iteration complexity of Algorithm 3, whose proof is given in [26, Theorem 2].

Theorem 2 (Complexity of Algorithm 3). Suppose that Assumption 4 holds. Let $H^{*}, H$ $D_{p}, D_{q}$, and $H_{\text {low }}$ be defined in (85), (86) and (87), $L_{\nabla h}$ be given in Assumption 4, $\epsilon$, $\epsilon_{0}$ and $x^{0}$ be given in Algorithm 3, and

$$
\begin{aligned}
\alpha= & \min \left\{1, \sqrt{4 \epsilon /\left(D_{q} L_{\nabla h}\right)}\right\}, \\
\delta= & \left(2+\alpha^{-1}\right) L_{\nabla h} D_{p}^{2}+\max \left\{\epsilon / D_{q}, \alpha L_{\nabla h} / 4\right\} D_{q}^{2}, \\
K= & {\left[16\left(\max _{y} H\left(x^{0}, y\right)-H^{*}+\epsilon D_{q} / 4\right) L_{\nabla h} \epsilon^{-2}+32 \epsilon_{0}^{2}\left(1+4 D_{q}^{2} L_{\nabla h}^{2} \epsilon^{-2}\right) \epsilon^{-2}-1\right]_{+}, } \\
N= & \left(\left[96 \sqrt{2}\left(1+\left(24 L_{\nabla h}+4 \epsilon / D_{q}\right) L_{\nabla h}^{-1}\right)\right]+2\right)\left\{2, \sqrt{D_{q} L_{\nabla h} \epsilon^{-1}}\right\} \\
& \times\left((K+1)\left(\log \frac{4 \max \left\{\frac{1}{2 L_{\nabla h}}, \min \left\{\frac{D_{q}}{\epsilon}, \frac{4}{\alpha L_{\nabla h}}\right\}\right\}\left(\delta+2 \alpha^{-1}\left(H^{*}-H_{\mathrm{low}}+\epsilon D_{q} / 4+L_{\nabla h} D_{p}^{2}\right)\right)}{\left[\left(3 L_{\nabla h}+\epsilon /\left(2 D_{q}\right)\right)^{2} / \min \left\{L_{\nabla h}, \epsilon /\left(2 D_{q}\right)\right\}+3 L_{\nabla h}+\epsilon /\left(2 D_{q}\right)\right]^{-2} \epsilon_{0}^{2}}\right)_{+}\right. \\
& +K+1+2 K \log (K+1)) .
\end{aligned}
$$

Then Algorithm 3 terminates and outputs an $\epsilon$-stationary point $\left(x_{\epsilon}, y_{\epsilon}\right)$ of (85) in at most $K+1$ outer iterations that satisfies

$$
\max _{y} H\left(x_{\epsilon}, y\right) \leq \max _{y} H\left(\hat{x}^{0}, y\right)+\epsilon D_{q} / 4+2 \epsilon_{0}^{2}\left(L_{\nabla h}^{-1}+4 D_{q}^{2} L_{\nabla h} \epsilon^{-2}\right)
$$

Moreover, the total number of evaluations of $\nabla h$ and proximal operator of $p$ and $q$ performed in Algorithm 3 is no more than $N$, respectively.


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    ${ }^{1}$ The definition of $L_{F}$-Lipschitz continuity of $F$ and $L_{\nabla f}$-smoothness of $f$ is given in Subsection

[^1]:    ${ }^{2}$ For convenience, $\infty$ stands for $+\infty$.

[^2]:    ${ }^{3}$ The latter part of this assumption can be weakened to the one that the pointwise Slater's condition holds for

[^3]:    the constraint on $y$ in (1), that is, there exists $\hat{y}_{x} \in \mathcal{Y}$ such that $d\left(x, \hat{y}_{x}\right)<0$ for each $x \in \mathcal{X}$. Indeed, if $\delta_{d}>0$, Assumption (3i) holds. Otherwise, one can solve the perturbed counterpart of (1) with $d(x, y)$ being replaced by $d(x, y)-\epsilon$ for some suitable $\epsilon>0$ instead, which satisfies Assumption (3i).

[^4]:    ${ }^{4}$ For convenience, $-\sigma_{x} \operatorname{dom} p$ stands for the set $\left\{-\sigma_{x} u \mid u \in \operatorname{dom} p\right\}$.

