A first-order augmented Lagrangian method for constrained minimax optimization

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Abstract

In this paper we study a class of constrained minimax problems. In particular, we propose a first-order augmented Lagrangian method for solving them, whose subproblems turn out to be a much simpler structured minimax problem and are suitably solved by a firstorder method recently developed in [26] by the authors. Under some suitable assumptions, an *operation complexity* of $\mathcal{O}(\varepsilon^{-4} \log \varepsilon^{-1})$, measured by its fundamental operations, is established for the first-order augmented Lagrangian method for finding an ε -KKT solution of the constrained minimax problems.

Keywords: minimax optimization, augmented Lagrangian method, first-order method, operation complexity

Mathematics Subject Classification: 90C26, 90C30, 90C47, 90C99, 65K05

1 Introduction

In this paper, we consider a constrained minimax problem

$$F^* = \min_{c(x) \le 0} \max_{d(x,y) \le 0} \{ F(x,y) := f(x,y) + p(x) - q(y) \}.$$
 (1)

Assume that problem (1) has at least one optimal solution and the following additional assumptions hold.

- Assumption 1. (i) F is L_F -Lipschitz continuous on $\mathcal{X} \times \mathcal{Y}$, f is $L_{\nabla f}$ -smooth on $\mathcal{X} \times \mathcal{Y}$, and $f(x, \cdot)$ is concave for any given $x \in \mathcal{X}$, where $\mathcal{X} := \text{dom } p$ and $\mathcal{Y} := \text{dom } q.^1$
- (ii) $p : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ and $q : \mathbb{R}^m \to \mathbb{R} \cup \{\infty\}$ are proper closed convex functions, and the proximal operator of p and q can be exactly evaluated.
- (iii) $c : \mathbb{R}^n \to \mathbb{R}^{\tilde{n}}$ is $L_{\nabla c}$ -smooth and L_c -Lipschitz continuous on \mathcal{X} , $d : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{\tilde{m}}$ is $L_{\nabla d}$ -smooth and L_d -Lipschitz continuous on $\mathcal{X} \times \mathcal{Y}$, and $d_i(x, \cdot)$ is convex for each $x \in \mathcal{X}$.
- (iv) The sets \mathcal{X} and \mathcal{Y} (namely, dom p and dom q) are compact.

In the recent years, the minimax problem of a simpler form

$$\min_{x \in X} \max_{y \in Y} f(x; y), \tag{2}$$

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¹The definition of L_F -Lipschitz continuity of F and $L_{\nabla f}$ -smoothness of f is given in Subsection 1.1.

where X and Y are a closed set, has received tremendous amount of attention. Indeed, it has found broad applications in many areas, such as adversarial training [16, 29, 40, 45], generative adversarial networks [13, 15, 37], reinforcement learning [8, 11, 31, 34, 41], computational game [1, 35, 42], distributed computing [30, 39], prediction and regression [4, 43, 49, 50], and distributionally robust optimization [12, 38]. Numerous methods have been developed for solving (2) with X and Y being a simple closed convex set (e.g., see [6, 18, 19, 22, 23, 25, 28, 33, 47, 51, 52, 54]).

There have also been several studies on some other special cases of problem (1) recently. In particular, two first-order methods, called max-oracle gradient-descent and nested gradient descent/ascent methods, were proposed in [14] for solving (1) with $c(x) \equiv 0$ and p and q being the indicator function of simple compact convex sets X and Y respectively, under the assumption that the function $V(x) = \max_{y \in Y} \{f(x, y) : d(x, y) \leq 0\}$ is convex and moreover an optimal Lagrangian multiplier associated with the constraint $d(x,y) \leq 0$ can be computed for each $x \in X$. In addition, a multiplier gradient descent method was proposed in [44] for solving (1) with $c(x) \equiv 0$, d(x, y) being an affine mapping, and p and q being the indicator function of a simple compact convex set. Also, a proximal gradient multi-step ascent decent method was developed in [9] for (1) with $c(x) \equiv 0$, d(x,y) being an affine mapping, and $f(x,y) = g(x) + x^T A y - h(y)$, under the assumption that f(x,y) - q(y) is strongly concave in y. Besides, primal dual alternating proximal gradient methods were proposed in [53] for (1)with $c(x) \equiv 0$, d(x,y) being an affine mapping, and $\{f(x,y) \text{ being strongly concave in } y \text{ or } (f(x,y)) \in [0,1], (f(x,y)) \in [f(x,y)) \in [$ $[q(y) \equiv 0 \text{ and } f(x, y) \text{ being a linear function in } y]$. For these methods, an iteration complexity for finding an approximate stationary point of the aforementioned special minimax problem was established in [9, 14, 53], respectively. Yet, their operation complexity, measured by the amount of fundamental operations such as evaluations of gradient of f and proximal operator of p and q, was not studied in these works.

There was no algorithmic development for (1) prior to our work, though optimality conditions of (1) were recently studied in [10]. In this paper, we propose a first-order augmented Lagrangian (AL) method for solving (1). Specifically, given an iterate (x^k, y^k) and a Lagrangian multiplier estimate $(\lambda_{\mathbf{x}}^k, \lambda_{\mathbf{y}}^k)$ at the *k*th iteration, the next iterate (x^{k+1}, y^{k+1}) is obtained by finding an approximate stationary point of the AL subproblem

$$\min_{x} \max_{y} \mathcal{L}(x, y, \lambda_{\mathbf{x}}^{k}, \lambda_{\mathbf{y}}^{k}; \rho_{k})$$

for some $\rho_k > 0$ through the use of a first-order method proposed in [26], where \mathcal{L} is the AL function of (1) defined as

$$\mathcal{L}(x, y, \lambda_{\mathbf{x}}, \lambda_{\mathbf{y}}; \rho) = F(x, y) + \frac{1}{2\rho} \left(\| [\lambda_{\mathbf{x}} + \rho c(x)]_{+} \|^{2} - \| \lambda_{\mathbf{x}} \|^{2} \right) - \frac{1}{2\rho} \left(\| [\lambda_{\mathbf{y}} + \rho d(x, y)]_{+} \|^{2} - \| \lambda_{\mathbf{y}} \|^{2} \right).$$
⁽³⁾

(3) The Lagrangian multiplier estimate is then updated by $\lambda_{\mathbf{x}}^{k+1} = \prod_{\mathbb{B}^+_{\Lambda}} (\lambda_{\mathbf{x}}^k + \rho_k c(x^{k+1}))$ and $\lambda_{\mathbf{y}}^{k+1} = [\lambda_{\mathbf{y}}^k + \rho_k d(x^{k+1}, y^{k+1})]_+$ for some $\Lambda > 0$, where $\prod_{\mathbb{B}^+_{\Lambda}} (\cdot)$ and $[\cdot]_+$ are defined in Section 1.1.

The main contributions of this paper are summarized below.

- We propose a first-order AL method for solving problem (1). To the best of our knowledge, this is the first yet implementable method for solving (1).
- We show that under some suitable assumptions, our first-order AL method enjoys an iteration complexity of $\mathcal{O}(\log \varepsilon^{-1})$ and an operation complexity of $\mathcal{O}(\varepsilon^{-4} \log \varepsilon^{-1})$, measured by the amount of evaluations of ∇f , ∇c , ∇d and proximal operator of p and q, for finding an ε -KKT solution of (1).

The rest of this paper is organized as follows. In Subsection 1.1, we introduce some notation and terminology. In Section 2, we propose a first-order AL method for solving problem (1). In Section 3, we present complexity results for the proposed method. In Section 4, we provide the proof of the main result.

1.1 Notation and terminology

The following notation will be used throughout this paper. Let \mathbb{R}^n denote the Euclidean space of dimension n and \mathbb{R}^n_+ denote the nonnegative orthant in \mathbb{R}^n . The standard inner product, l_1 -norm and Euclidean norm are denoted by $\langle \cdot, \cdot \rangle$, $\|\cdot\|_1$ and $\|\cdot\|$, respectively. For any $\Lambda > 0$, let $\mathbb{B}^+_{\Lambda} = \{x \ge 0 : \|x\| \le \Lambda\}$, whose dimension is clear from the context. For any $v \in \mathbb{R}^n$, let v_+ denote the nonnegative part of v, that is, $(v_+)_i = \max\{v_i, 0\}$ for all i. Given a point x and a closed set S in \mathbb{R}^n , let dist $(x, S) = \min_{x' \in S} \|x' - x\|$, $\Pi_S(x)$ denote the Euclidean projection of x onto S, and \mathscr{I}_S denote the indicator function associated with S.

A function or mapping ϕ is said to be L_{ϕ} -Lipschitz continuous on a set S if $\|\phi(x) - \phi(x')\| \le L_{\phi} \|x - x'\|$ for all $x, x' \in S$. In addition, it is said to be $L_{\nabla \phi}$ -smooth on S if $\|\nabla \phi(x) - \nabla \phi(x')\| \le L_{\nabla \phi} \|x - x'\|$ for all $x, x' \in S$. For a closed convex function $p : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\},^2$ the proximal operator associated with p is denoted by prox_n , that is,

$$\operatorname{prox}_p(x) = \arg\min_{x' \in \mathbb{R}^n} \left\{ \frac{1}{2} \|x' - x\|^2 + p(x') \right\} \quad \forall x \in \mathbb{R}^n.$$

Given that evaluation of $\operatorname{prox}_{\gamma p}(x)$ is often as cheap as $\operatorname{prox}_p(x)$, we count the evaluation of $\operatorname{prox}_{\gamma p}(x)$ as one evaluation of proximal operator of p for any $\gamma > 0$ and $x \in \mathbb{R}^n$.

For a lower semicontinuous function $\phi : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$, its *domain* is the set dom $\phi := \{x | \phi(x) < \infty\}$. The *upper subderivative* of ϕ at $x \in \text{dom } \phi$ in a direction $d \in \mathbb{R}^n$ is defined by

$$\phi'(x;d) = \limsup_{x' \stackrel{\phi}{\to} x, t \downarrow 0} \inf_{d' \to d} \frac{\phi(x' + td') - \phi(x')}{t},$$

where $t \downarrow 0$ means both t > 0 and $t \to 0$, and $x' \stackrel{\phi}{\to} x$ means both $x' \to x$ and $\phi(x') \to \phi(x)$. The subdifferential of ϕ at $x \in \text{dom } \phi$ is the set

$$\partial \phi(x) = \{ s \in \mathbb{R}^n | s^T d \le \phi'(x; d) \; \forall d \in \mathbb{R}^n \}.$$

We use $\partial_{x_i}\phi(x)$ to denote the subdifferential with respect to x_i . In addition, for an upper semicontinuous function ϕ , its subdifferential is defined as $\partial \phi = -\partial(-\phi)$. If ϕ is locally Lipschitz continuous, the above definition of subdifferential coincides with the Clarke subdifferential. Besides, if ϕ is convex, it coincides with the ordinary subdifferential for convex functions. Also, if ϕ is continuously differentiable at x, we simply have $\partial \phi(x) = \{\nabla \phi(x)\}$, where $\nabla \phi(x)$ is the gradient of ϕ at x. In addition, it is not hard to verify that $\partial(\phi_1 + \phi_2)(x) = \nabla \phi_1(x) + \partial \phi_2(x)$ if ϕ_1 is continuously differentiable at x and ϕ_2 is lower or upper semicontinuous at x. See [7, 46] for more details.

Finally, we introduce an (approximate) stationary point (e.g., see [9, 10, 21]) for a general minimax problem

$$\min_{x} \max_{y} \Psi(x, y), \tag{4}$$

where $\Psi(\cdot, y) : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is a lower semicontinuous function, and $\Psi(x, \cdot) : \mathbb{R}^m \to \mathbb{R} \cup \{-\infty\}$ is an upper semicontinuous function.

Definition 1. A point (x, y) is said to be a stationary point of the minimax problem (4) if

 $0 \in \partial_x \Psi(x, y), \quad 0 \in \partial_y \Psi(x, y).$

In addition, for any $\epsilon > 0$, a point $(x_{\epsilon}, y_{\epsilon})$ is said to be an ϵ -stationary point of the minimax problem (4) if

$$\operatorname{dist}\left(0,\partial_x\Psi(x_{\epsilon},y_{\epsilon})\right) \leq \epsilon, \quad \operatorname{dist}\left(0,\partial_y\Psi(x_{\epsilon},y_{\epsilon})\right) \leq \epsilon.$$

²For convenience, ∞ stands for $+\infty$.

2 A first-order augmented Lagrangian method for problem (1)

In this section we propose a first-order augmented Lagrangian (FAL) method for problem (1).

One standard approach for solving constrained nonlinear program is to solve a sequence of unconstrained nonlinear program problems, which are typically penalty or augmented Lagrangian subproblems (e.g., see [32]). In a similar spirit, we next propose an FAL method in Algorithm 1 for solving (1). In particular, at each iteration, the FAL method finds an approximate stationary point of an AL subproblem in the form of

$$\min_{x} \max_{y} \mathcal{L}(x, y, \lambda_{\mathbf{x}}, \lambda_{\mathbf{y}}; \rho)$$
(5)

for some $\rho > 0$, $\lambda_{\mathbf{x}} \in \mathbb{R}^{\tilde{n}}_{+}$ and $\lambda_{\mathbf{y}} \in \mathbb{R}^{\tilde{m}}_{+}$, where \mathcal{L} is the AL function associated with problem (1) defined in (3). In view of Assumption 1, one can observe that \mathcal{L} enjoys the following nice properties.

- For any given $\rho > 0$, $\lambda_{\mathbf{x}} \in \mathbb{R}^{\tilde{n}}_{+}$ and $\lambda_{\mathbf{y}} \in \mathbb{R}^{\tilde{m}}_{+}$, \mathcal{L} is the sum of smooth function $f(x, y) + (\|[\lambda_{\mathbf{x}} + \rho c(x)]_{+}\|^{2} \|\lambda_{\mathbf{x}}\|^{2})/(2\rho) (\|[\lambda_{\mathbf{y}} + \rho d(x, y)]_{+}\|^{2} \|\lambda_{\mathbf{y}}\|^{2})/(2\rho)$ with Lipschitz continuous gradient and possibly nonsmooth function p(x) q(y) with exactly computable proximal operator.
- \mathcal{L} is nonconvex in x but concave in y.

Thanks to such a nice structure of \mathcal{L} , an approximate stationary point of the AL subproblem (5) can be found by Algorithm 3 (see Appendix A), which is a first-order method proposed in [26, Algorithm 2]) for solving nonconvex-concave minimax problems.

Before presenting an FAL method for (1), we let

$$\mathcal{L}_{\mathbf{x}}(x,y,\lambda_{\mathbf{x}};\rho) := F(x,y) + \frac{1}{2\rho} \left(\| [\lambda_{\mathbf{x}} + \rho c(x)]_{+} \|^{2} - \|\lambda_{\mathbf{x}}\|^{2} \right), \tag{6}$$

$$c_{\rm hi} := \max\{\|c(x)\| | x \in \mathcal{X}\}, \quad d_{\rm hi} := \max\{\|d(x,y)\| | (x,y) \in \mathcal{X} \times \mathcal{Y}\},\tag{7}$$

and make one additional assumption on problem (1).

Assumption 2. For any given $\eta \in (0, 1]$, an η -approximately feasible point z_{η} of problem (1), namely $z_{\eta} \in \mathcal{X}$ satisfying $||[c(z_{\eta})]_{+}|| \leq \eta$, can be found.

Remark 1. A very similar assumption as Assumption 2 was considered in [5, 17, 27, 48]. One example of the problem instances satisfying Assumption 2 arises when the error bound condition $\|[c(x)]_+\| = \mathcal{O}(\operatorname{dist}(0,\partial(\|[c(x)]_+\|^2 + \mathscr{I}_{\mathcal{X}}(x))))^{\nu})$ holds on a level set of $\|[c(x)]_+\|$ for some $\nu > 0$ (e.g., see [24, 36]). Indeed, one can find the above z_{η} by applying a projected gradient method to the problem $\min_{x \in \mathcal{X}} \|[c(x)]_+\|^2$.

We are now ready to present an FAL method for solving problem (1).

Algorithm 1 A first-order augmented Lagrangian method for problem (1)

Input: $\varepsilon, \tau \in (0, 1), \epsilon_0 \in (\tau \varepsilon, 1], \epsilon_k = \epsilon_0 \tau^k, \rho_k = \epsilon_k^{-1}, \Lambda > 0, \lambda_{\mathbf{x}}^0 \in \mathbb{B}^+_{\Lambda}, \lambda_{\mathbf{y}}^0 \in \mathbb{R}^{\tilde{m}}_+, (x^0, y^0) \in \mathcal{X} \times \mathcal{Y}, \text{ and } x_{\mathbf{nf}} \in \mathcal{X} \text{ with } \|[c(x_{\mathbf{nf}})]_+\| \leq \sqrt{\varepsilon}.$ 1: **for** $k = 0, 1, \ldots$ **do** 2: Set

$$x_{\text{init}}^{k} = \begin{cases} x^{k}, & \text{if } \mathcal{L}_{\mathbf{x}}(x^{k}, y^{k}, \lambda_{\mathbf{x}}^{k}; \rho_{k}) \leq \mathcal{L}_{\mathbf{x}}(x_{\mathbf{nf}}, y^{k}, \lambda_{\mathbf{x}}^{k}; \rho_{k}), \\ x_{\mathbf{nf}}, & \text{otherwise.} \end{cases}$$
(8)

3: Call Algorithm 3 (see Appendix A) with $\epsilon \leftarrow \epsilon_k$, $\epsilon_0 \leftarrow \epsilon_k/(2\sqrt{\rho_k})$, $(x^0, y^0) \leftarrow (x^k_{\text{init}}, y^k)$ and $L_{\nabla h} \leftarrow L_k$ to find an ϵ_k -stationary point (x^{k+1}, y^{k+1}) of

$$\min_{x} \max_{y} \mathcal{L}(x, y, \lambda_{\mathbf{x}}^{k}, \lambda_{\mathbf{y}}^{k}; \rho_{k})$$
(9)

where

$$L_k = L_{\nabla f} + \rho_k L_c^2 + \rho_k c_{\rm hi} L_{\nabla c} + \|\lambda_{\mathbf{x}}^k\| L_{\nabla c} + \rho_k L_d^2 + \rho_k d_{\rm hi} L_{\nabla d} + \|\lambda_{\mathbf{y}}^k\| L_{\nabla d}.$$
 (10)

- 4: Set $\lambda_{\mathbf{x}}^{k+1} = \prod_{\mathbb{B}^+_{\lambda}} (\lambda_{\mathbf{x}}^k + \rho_k c(x^{k+1}))$ and $\lambda_{\mathbf{y}}^{k+1} = [\lambda_{\mathbf{y}}^k + \rho_k d(x^{k+1}, y^{k+1})]_+$.
- 5: Terminate the algorithm and output (x^{k+1}, y^{k+1}) if $\epsilon_k \leq \varepsilon$.

6: end for

- **Remark 2.** (i) x_{nf} is an $\sqrt{\varepsilon}$ -approximately feasible point of problem (1), where the subscript "nf" stands for "nearly feasible". It follows from Assumption 2 that x_{nf} can be found in advance.
 - (ii) $\lambda_{\mathbf{x}}^{k+1}$ results from projecting onto a nonnegative Euclidean ball the standard Lagrangian multiplier estimate $\tilde{\lambda}_{\mathbf{x}}^{k+1}$ obtained by the classical scheme $\tilde{\lambda}_{\mathbf{x}}^{k+1} = [\lambda_{\mathbf{x}}^{k} + \rho_{k}c(x^{k+1})]_{+}$. It is called a safeguarded Lagrangian multiplier in the relevant literature [2, 20, 3], which has been shown to enjoy many practical and theoretical advantages (see [2] for discussions).
- (iii) In view of Theorem 2 (see Appendix A), one can see that an ϵ_k -stationary point of (9) can be successfully found in step 3 of Algorithm 1 by applying Algorithm 3 to problem (9) and thus Algorithm 1 is well-defined.

3 Complexity results of Algorithm 1

In this section we establish iteration and operation complexity results for Algorithm 1. Before proceeding, we make one additional assumption that a generalized Mangasarian-Fromowitz constraint qualification holds for the minimization part of (1) and a uniform Slater's condition holds for the maximization part of (1).

Assumption 3. (i) There exist some constants δ_c , θ_a , $\theta_f > 0$ such that for each $x \in \mathcal{F}(\theta_f)$ there exists some $v_x \in \mathbb{R}^n$ satisfying $||v_x|| = 1$ and $v_x^T \nabla c_i(x) \leq -\delta_c$ for all $i \in \mathcal{A}(x; \theta_a)$, where

$$\mathcal{F}(\theta_f) = \{ x \in \mathcal{X} \big| \| [c(x)]_+ \| \le \theta_f \}, \quad \mathcal{A}(x; \theta_a) = \{ i | c_i(x) \ge -\theta_a, \ 1 \le i \le \tilde{n} \}.$$
(11)

(ii) For each $x \in \mathcal{X}$, there exists some $\hat{y}_x \in \mathcal{Y}$ such that $d_i(x, \hat{y}_x) < 0$ for all $i = 1, 2, ..., \tilde{m}$, and moreover, $\delta_d := \inf\{-d_i(x, \hat{y}_x) | x \in \mathcal{X}, i = 1, 2, ..., \tilde{m}\} > 0.^3$

³The latter part of this assumption can be weakened to the one that the pointwise Slater's condition holds for

In order to characterize the approximate solution found by Algorithm 1, we next introduce a terminology called an ε -KKT solution of problem (1).

One can observe from Lemma 1(iii) that problem (1) is equivalent to

$$\min_{x,\lambda_{\mathbf{y}}} \left\{ \max_{y} F(x,y) - \langle \lambda_{\mathbf{y}}, d(x,y) \rangle + \mathscr{I}_{\mathbb{R}^{\tilde{m}}_{+}}(\lambda_{\mathbf{y}}) \Big| c(x) \le 0 \right\}.$$

By this, one can further see that problem (1) is equivalent to

$$\min_{x,\lambda_{\mathbf{y}}} \max_{\lambda_{\mathbf{x}}} \Big\{ \max_{y} \{ F(x,y) - \langle \lambda_{\mathbf{y}}, d(x,y) \rangle + \mathscr{I}_{\mathbb{R}^{\tilde{m}}_{+}}(\lambda_{\mathbf{y}}) \} + \langle \lambda_{\mathbf{x}}, c(x) \rangle - \mathscr{I}_{\mathbb{R}^{\tilde{n}}_{+}}(\lambda_{\mathbf{x}}) \Big\},$$

which is a nonconvex-concave minimax problem

$$\min_{x,\lambda_{\mathbf{y}}} \max_{y,\lambda_{\mathbf{x}}} \left\{ F(x,y) + \langle \lambda_{\mathbf{x}}, c(x) \rangle - \langle \lambda_{\mathbf{y}}, d(x,y) \rangle - \mathscr{I}_{\mathbb{R}^{\tilde{n}}_{+}}(\lambda_{\mathbf{x}}) + \mathscr{I}_{\mathbb{R}^{\tilde{m}}_{+}}(\lambda_{\mathbf{y}}) \right\}.$$
(12)

It then follows from Definition 1 (see also [9, Theorem 3]) that $(x, y, \lambda_{\mathbf{x}}, \lambda_{\mathbf{y}}) \in \mathbb{R}^n \times \mathbb{R}^{\tilde{n}} \times \mathbb{R}^{\tilde{n}}_+ \times \mathbb{R}^{\tilde{m}}_+$ is a stationary point of problem (12) if

$$0 \in \partial_x F(x, y) + \nabla c(x)\lambda_{\mathbf{x}} - \nabla_x d(x, y)\lambda_{\mathbf{y}}, \tag{13}$$

$$0 \in \partial_y F(x, y) - \nabla_y d(x, y) \lambda_{\mathbf{y}},\tag{14}$$

$$c(x) \le 0, \quad \langle \lambda_{\mathbf{x}}, c(x) \rangle = 0,$$
(15)

$$d(x,y) \le 0, \quad \langle \lambda_{\mathbf{v}}, d(x,y) \rangle = 0. \tag{16}$$

Based on this observation and the equivalence of (1) and (12), we introduce an (approximate) KKT solution of problem (1) below.

Definition 2. The pair (x, y) is said to be a KKT solution of problem (1) if there exists $(\lambda_{\mathbf{x}}, \lambda_{\mathbf{y}}) \in \mathbb{R}^{\tilde{n}}_{+} \times \mathbb{R}^{\tilde{m}}_{+}$ such that the conditions (13)-(16) hold. In addition, for any $\varepsilon > 0$, (x, y) is said to be an ε -KKT point of problem (1) if there exists $(\lambda_{\mathbf{x}}, \lambda_{\mathbf{y}}) \in \mathbb{R}^{\tilde{n}}_{+} \times \mathbb{R}^{\tilde{m}}_{+}$ such that

$$\begin{aligned} \operatorname{dist}(0, \partial_x F(x, y) + \nabla c(x)\lambda_{\mathbf{x}} - \nabla_x d(x, y)\lambda_{\mathbf{y}}) &\leq \varepsilon, \\ \operatorname{dist}(0, \partial_y F(x, y) - \nabla_y d(x, y)\lambda_{\mathbf{y}}) &\leq \varepsilon, \\ \|[c(x)]_+\| &\leq \varepsilon, \quad |\langle \lambda_{\mathbf{x}}, c(x) \rangle| &\leq \varepsilon, \\ \|[d(x, y)]_+\| &\leq \varepsilon, \quad |\langle \lambda_{\mathbf{y}}, d(x, y) \rangle| &\leq \varepsilon. \end{aligned}$$

To study complexity of Algorithm 1, we define

$$f^*(x) := \max\{F(x,y) | d(x,y) \le 0\},\tag{17}$$

$$f_{\text{low}}^* := \inf\{f^*(x) | x \in \mathcal{X}\},\tag{18}$$

$$D_{\mathbf{x}} \coloneqq \max\{\|u - v\| | u, v \in \mathcal{X}\}, \quad D_{\mathbf{y}} \coloneqq \max\{\|u - v\| | u, v \in \mathcal{Y}\},$$
(19)

$$F_{\rm hi} := \max\{F(x,y)|(x,y) \in \mathcal{X} \times \mathcal{Y}\}, \quad F_{\rm low} := \min\{F(x,y)|(x,y) \in \mathcal{X} \times \mathcal{Y}\}, \tag{20}$$

$$r := 2\delta_d^{-1}(\epsilon_0 + L_F)D_{\mathbf{y}},\tag{21}$$

$$K := \left[(\log \varepsilon - \log \epsilon_0) / \log \tau \right]_+, \quad \mathbb{K} := \{0, 1, \dots, K+1\},$$
(22)

where L_F and δ_d are given in Assumptions 1 and 3, and ϵ_0 , ε , and τ are some input parameters of Algorithm 1. For convenience, we define $\mathbb{K} - 1 = \{k - 1 | k \in \mathbb{K}\}$. One can observe from Assumption 1 that $D_{\mathbf{x}}$, $D_{\mathbf{y}}$, F_{hi} and F_{low} are finite. Besides, as will be shown in Lemma 1, f_{low}^* is also finite.

We are now ready to present an *iteration and operation complexity* of Algorithm 1 for finding an $\mathcal{O}(\varepsilon)$ -KKT solution of problem (1), whose proof is deferred to Section 4.

the constraint on y in (1), that is, there exists $\hat{y}_x \in \mathcal{Y}$ such that $d(x, \hat{y}_x) < 0$ for each $x \in \mathcal{X}$. Indeed, if $\delta_d > 0$, Assumption 3(ii) holds. Otherwise, one can solve the perturbed counterpart of (1) with d(x, y) being replaced by $d(x, y) - \epsilon$ for some suitable $\epsilon > 0$ instead, which satisfies Assumption 3(ii).

Theorem 1. Suppose that Assumptions 1, 2 and 3 hold. Let $\{(x^k, y^k, \lambda_{\mathbf{x}}^k, \lambda_{\mathbf{y}}^k)\}_{k \in \mathbb{K}}$ be generated by Algorithm 1, c_{hi} , d_{hi} , f_{low}^* , $D_{\mathbf{x}}$, $D_{\mathbf{y}}$, F_{hi} , F_{low} and K be defined in (7), (18), (19), (20) and (22), L_F , $L_{\nabla f}$, $L_{\nabla d}$, $L_{\nabla c}$, L_c , $L_{\nabla d}$, L_d and δ_d be given in Assumption 1, ε , ϵ_0 , τ , Λ and $\lambda_{\mathbf{y}}^0$ be given in Algorithm 1, and

$$\widehat{L} = L_{\nabla f} + L_c^2 + c_{\rm hi} L_{\nabla c} + \Lambda L_{\nabla c} + L_d^2 + d_{\rm hi} L_{\nabla d} + L_{\nabla d} \sqrt{\|\lambda_{\mathbf{y}}^0\|^2 + \frac{2(F_{\rm hi} - f_{\rm low}^* + D_{\mathbf{y}}\epsilon_0)}{1 - \tau}}, \qquad (23)$$

$$\hat{\alpha} = \min\left\{1, \sqrt{4/(D_{\mathbf{y}}\hat{L})}\right\}, \quad \hat{\delta} = (2 + \hat{\alpha}^{-1})\hat{L}D_{\mathbf{x}}^2 + \max\{1/D_{\mathbf{y}}, \hat{L}/4\}D_{\mathbf{y}}^2, \tag{24}$$

$$\widehat{M} = 16 \max\left\{ \frac{1}{(2L_c^2)}, \frac{4}{(\hat{\alpha}L_c^2)} \right\} \left[(3\widehat{L} + \frac{1}{(2D_y)})^2 / \min\{L_c^2, \frac{1}{(2D_y)} + 3\widehat{L} + \frac{1}{(2D_y)} \right]^2 \\ \times \left(\hat{\delta} + 2\hat{\alpha}^{-1} \left(F_{\rm hi} - F_{\rm low} + \frac{\Lambda^2}{2} + \frac{3}{2} \|\lambda_y^0\|^2 + \frac{3(F_{\rm hi} - f_{\rm low}^* + D_y\epsilon_0)}{1 - \tau} + \rho_k d_{\rm hi}^2 + \frac{D_y}{4} + \widehat{L}D_x^2 \right) \right]$$
(25)

$$\widehat{T} = \left[16 \left(L_F D_{\mathbf{y}} + F_{\rm hi} - f_{\rm low}^* + \Lambda + \frac{1}{2} (\tau^{-1} + \|\lambda_{\mathbf{y}}^0\|^2) + \frac{F_{\rm hi} - f_{\rm low}^* + D_{\mathbf{y}} \epsilon_0}{1 - \tau} + \frac{\Lambda^2}{2} + \frac{D_{\mathbf{y}}}{4} \right) \widehat{L} + 8(1 + 4D_{\mathbf{y}}^2 \widehat{L}^2) \right]_+,$$
(26)

$$\tilde{\lambda}_{\mathbf{x}}^{K+1} = [\lambda_{\mathbf{x}}^{K} + c(x^{K+1})/(\epsilon_0 \tau^K)]_+.$$
(27)

Suppose that

$$\varepsilon^{-1} \ge \max\left\{1, \theta_{a}^{-1}\Lambda, \theta_{f}^{-2}\left\{2L_{F}D_{\mathbf{y}} + 2F_{\mathrm{hi}} - 2f_{\mathrm{low}}^{*} + 2\Lambda + \tau^{-1} + \|\lambda_{\mathbf{y}}^{0}\|^{2} + \frac{2(F_{\mathrm{hi}} - f_{\mathrm{low}}^{*} + D_{\mathbf{y}}\epsilon_{0})}{1 - \tau} + \frac{\epsilon_{0}D_{\mathbf{y}}}{2} + L_{c}^{-2} + 4D_{\mathbf{y}}^{2}\widehat{L} + \Lambda^{2}\right\}, \frac{4\|\lambda_{\mathbf{y}}^{0}\|^{2}}{\delta_{d}^{2}\tau} + \frac{8(F_{\mathrm{hi}} - f_{\mathrm{low}}^{*} + D_{\mathbf{y}}\epsilon_{0})}{\delta_{d}^{2}\tau(1 - \tau)}\right\}.$$
(28)

Then the following statements hold.

(i) Algorithm 1 terminates after K+1 outer iterations and outputs an approximate stationary point (x^{K+1}, y^{K+1}) of (1) satisfying

$$\operatorname{dist}(0, \partial_x F(x^{K+1}, y^{K+1}) + \nabla c(x^{K+1}) \tilde{\lambda}_x^{K+1} - \nabla_x d(x^{K+1}, y^{K+1}) \lambda_y^{K+1}) \le \varepsilon,$$
(29)

dist
$$(0, \partial_y F(x^{K+1}, y^{K+1}) - \nabla_y d(x^{K+1}, y^{K+1}) \lambda_{\mathbf{y}}^{K+1}) \le \varepsilon,$$
 (30)

$$\|[c(x^{K+1})]_{+}\| \leq \varepsilon \delta_{c}^{-1} \left(L_{F} + 2L_{d} \delta_{d}^{-1} (\epsilon_{0} + L_{F}) D_{\mathbf{y}} + \epsilon_{0} \right),$$

$$|\langle \tilde{\lambda}_{\mathbf{x}}^{K+1}, c(x^{K+1}) \rangle| \leq \varepsilon \delta_{c}^{-1} (L_{F} + 2L_{d} \delta_{d}^{-1} (\epsilon_{0} + L_{F}) D_{\mathbf{y}} + \epsilon_{0})$$

$$(31)$$

$$|\langle c(x^{n+1})\rangle| \leq \varepsilon \delta_c^{-1} (L_F + 2L_d \delta_d^{-1} (\epsilon_0 + L_F) D_{\mathbf{y}} + \epsilon_0) \\ \times \max\{\delta_c^{-1} (L_F + 2L_d \delta_d^{-1} (\epsilon_0 + L_F) D_{\mathbf{y}} + \epsilon_0), \Lambda\},$$
(32)

$$\|[d(x^{K+1}, y^{K+1})]_+\| \le 2\varepsilon \delta_d^{-1}(\epsilon_0 + L_F) D_{\mathbf{y}},\tag{33}$$

$$|\langle \lambda_{\mathbf{y}}^{K+1}, d(x^{K+1}, y^{K+1}) \rangle| \le 2\varepsilon \delta_d^{-1}(\epsilon_0 + L_F) D_{\mathbf{y}} \max\{2\delta_d^{-1}(\epsilon_0 + L_F) D_{\mathbf{y}}, \|\lambda_{\mathbf{y}}^0\|\}$$
(34)

(ii) The total number of evaluations of ∇f , ∇c , ∇d and proximal operator of p and q performed in Algorithm 1 is at most N, respectively, where

$$N = \left(\left\lceil 96\sqrt{2} \left(1 + \left(24\widehat{L} + 4/D_{\mathbf{y}} \right) / L_c^2 \right) \right\rceil + 2 \right) \max \left\{ 2, \sqrt{D_{\mathbf{y}}\widehat{L}} \right\} \widehat{T} (1 - \tau^4)^{-1} \\ \times (\tau \varepsilon)^{-4} \left(28K \log(1/\tau) + 28 \log(1/\epsilon_0) + 2(\log \widehat{M})_+ + 2 + 2\log(2\widehat{T}) \right).$$
(35)

Remark 3. One can observe from Theorem 1 that Algorithm 1 enjoys an iteration complexity of $\mathcal{O}(\log \varepsilon^{-1})$ and an operation complexity of $\mathcal{O}(\varepsilon^{-4}\log \varepsilon^{-1})$, measured by the amount of evaluations of ∇f , ∇c , ∇d and proximal operator of p and q, for finding an $\mathcal{O}(\varepsilon)$ -KKT solution (x^{K+1}, y^{K+1}) of (1) such that

$$\begin{aligned} \operatorname{dist}\left(\partial_{x}F(x^{K+1}, y^{K+1}) + \nabla c(x^{K+1})\tilde{\lambda}_{\mathbf{x}} - \nabla_{x}d(x^{K+1}, y^{K+1})\lambda_{\mathbf{y}}^{K+1}\right) &\leq \varepsilon \\ \operatorname{dist}\left(\partial_{y}F(x^{K+1}, y^{K+1}) - \nabla_{y}d(x^{K+1}, y^{K+1})\lambda_{\mathbf{y}}^{K+1}\right) &\leq \varepsilon, \\ \|[c(x^{K+1})]_{+}\| &= \mathcal{O}(\varepsilon), \quad |\langle \tilde{\lambda}_{\mathbf{x}}^{K+1}, c(x^{K+1})\rangle| = \mathcal{O}(\varepsilon), \\ \|[d(x^{K+1}, y^{K+1})]_{+}\| &= \mathcal{O}(\varepsilon), \quad |\langle \lambda_{\mathbf{y}}^{K+1}, d(x^{K+1}, y^{K+1})\rangle| = \mathcal{O}(\varepsilon). \end{aligned}$$

where $\tilde{\lambda}_{\mathbf{x}}^{K+1} \in \mathbb{R}_{+}^{\tilde{n}}$ is defined in (27) and $\lambda_{\mathbf{y}}^{K+1} \in \mathbb{R}_{+}^{\tilde{m}}$ is given in Algorithm 1.

4 Proof of the main result

In this section, we provide a proof of our main result presented in Section 2, which is particularly Theorem 1. Before proceeding, let

$$\mathcal{L}_{\mathbf{y}}(x, y, \lambda_{\mathbf{y}}; \rho) = F(x, y) - \frac{1}{2\rho} \left(\| [\lambda_{\mathbf{y}} + \rho d(x, y)]_{+} \|^{2} - \| \lambda_{\mathbf{y}} \|^{2} \right).$$
(36)

In view of (3), (17) and (36), one can observe that

$$f^*(x) \le \max_{y} \mathcal{L}_{\mathbf{y}}(x, y, \lambda_{\mathbf{y}}; \rho) \qquad \forall x \in \mathcal{X}, \ \lambda_{\mathbf{y}} \in \mathbb{R}^{\tilde{m}}_+, \ \rho > 0,$$
(37)

which will be frequently used later.

We next establish several lemmas that will be used to prove Theorem 1 subsequently.

Lemma 1. Suppose that Assumptions 1 and 3 hold. Let f^* , f^*_{low} , D_y , r, L_F and δ_d be given in (17), (18), (19), (21) and Assumption 1, respectively. Then the following statements hold.

- (i) $\|\lambda_{\mathbf{y}}^*\| \leq \delta_d^{-1} L_F D_{\mathbf{y}}$ and $\lambda_{\mathbf{y}}^* \in \mathbb{B}_r^+$ for all $\lambda_{\mathbf{y}}^* \in \Lambda^*(x)$ and $x \in \mathcal{X}$, where $\Lambda^*(x)$ denotes the set of optimal Lagrangian multipliers of problem (17) for any $x \in \mathcal{X}$.
- (ii) The function f^* is Lipschitz continuous on \mathcal{X} and f^*_{low} is finite.
- (iii) It holds that

$$f^*(x) = \min_{\lambda_{\mathbf{y}}} \max_{y} F(x, y) - \langle \lambda_{\mathbf{y}}, d(x, y) \rangle + \mathscr{I}_{\mathbb{R}^{\tilde{m}}_+}(\lambda_{\mathbf{y}}) \qquad \forall x \in \mathcal{X},$$
(38)

where $\mathscr{I}_{\mathbb{R}^{\tilde{m}}_{+}}(\cdot)$ is the indicator function associated with $\mathbb{R}^{\tilde{m}}_{+}$.

Proof. (i) Let $x \in \mathcal{X}$ and $\lambda_{\mathbf{y}}^* \in \Lambda^*(x)$ be arbitrarily chosen, and let $y^* \in \mathcal{Y}$ be such that $(y^*, \lambda_{\mathbf{y}}^*)$ is a pair of primal-dual optimal solutions of (17). It then follows that

$$y^* \in \operatorname*{Argmax}_y F(x,y) - \langle \lambda^*_{\mathbf{y}}, d(x,y) \rangle, \quad \langle \lambda^*_{\mathbf{y}}, d(x,y^*) \rangle = 0, \quad d(x,y^*) \le 0, \quad \lambda^*_{\mathbf{y}} \ge 0.$$

The first relation above yields

$$F(x, y^*) - \langle \lambda_{\mathbf{y}}^*, d(x, y^*) \rangle \ge F(x, \hat{y}_x) - \langle \lambda_{\mathbf{y}}^*, d(x, \hat{y}_x) \rangle,$$

where \hat{y}_x is given in Assumption 3(ii). By this and $\langle \lambda_y^*, d(x, y^*) \rangle = 0$, one has

$$\langle \lambda_{\mathbf{y}}^*, -d(x, \hat{y}_x) \rangle \leq F(x, y^*) - F(x, \hat{y}_x),$$

which together with (19), $\lambda_{\mathbf{v}}^* \geq 0$ and Assumption 1 implies that

$$\delta_d \|\lambda_{\mathbf{y}}^*\|_1 \le \langle \lambda_{\mathbf{y}}^*, -d(x, \hat{y}_x) \rangle \le F(x, y^*) - F(x, \hat{y}_x) \le L_F \|y^* - \hat{y}_x\| \le L_F D_{\mathbf{y}}, \tag{39}$$

where the first inequality is due to Assumption 3(ii), and the third inequality follows from (19) and L_F -Lipschitz continuity of F (see Assumption 1(i)). Using (21) and (39), we have $\|\lambda_{\mathbf{y}}^*\| \leq \|\lambda_{\mathbf{y}}^*\|_1 \leq \delta_d^{-1} L_F D_{\mathbf{y}}$ and hence $\lambda_{\mathbf{y}}^* \in \mathbb{B}_r^+$ due to (21).

(ii) Recall from Assumption 1 that $F(x, \cdot)$ and $d_i(x, \cdot)$, $i = 1, \ldots, l$, are convex for any given $x \in \mathcal{X}$. Using this, (17), (21) and the first statement of this lemma, we observe that

$$f^*(x) = \max_{y} \min_{\lambda \in \mathbb{B}_r^+} F(x, y) - \langle \lambda, d(x, y) \rangle \qquad \forall x \in \mathcal{X}.$$
(40)

Notice from Assumption 1 that F and d are Lipschitz continuous on their domain. Then it is not hard to observe that $\min\{F(x, y) + \langle \lambda, d(x, y) \rangle | \lambda \in \mathbb{B}_r^+\}$ is a Lipschitz continuous function of (x, y) on its domain. By this and (40), one can easily verify that f^* is Lipschitz continuous on \mathcal{X} . In addition, the finiteness of f_{low}^* follows from (18), the continuity of \tilde{f}^* , and the compactness of \mathcal{X} .

(iii) One can observe from (17) that for all $x \in \mathcal{X}$,

$$f^*(x) = \max_{y} \min_{\lambda_{\mathbf{y}}} F(x, y) - \langle \lambda_{\mathbf{y}}, d(x, y) \rangle + \mathscr{I}_{\mathbb{R}^{\tilde{m}}_+}(\lambda_{\mathbf{y}}) \leq \min_{\lambda_{\mathbf{y}}} \max_{y} F(x, y) - \langle \lambda_{\mathbf{y}}, d(x, y) \rangle + \mathscr{I}_{\mathbb{R}^{\tilde{m}}_+}(\lambda_{\mathbf{y}}),$$

where the inequality follows from the weak duality. In addition, it follows from Assumption 1 that the domain of $F(x, \cdot)$ is compact for all $x \in \mathcal{X}$. By this, (40) and the strong duality, one has

$$f^*(x) = \min_{\lambda \in \mathbb{B}^+_r} \max_y F(x, y) - \langle \lambda, d(x, y) \rangle \quad \forall x \in \mathcal{X},$$

which together with the above inequality implies that (38) holds.

Lemma 2. Suppose that Assumptions 1 and 3 hold. Let $\{\lambda_{\mathbf{y}}^k\}_{k \in \mathbb{K}}$ be generated by Algorithm 1, f_{low}^* , $D_{\mathbf{y}}$, and F_{hi} be defined in (18), (19) and (20), and ϵ_0 , τ , and ρ_k be given in Algorithm 1. Then we have

$$\rho_{k}^{-1} \|\lambda_{\mathbf{y}}^{k}\|^{2} \leq \|\lambda_{\mathbf{y}}^{0}\|^{2} + \frac{2(F_{\text{hi}} - f_{\text{low}}^{*} + D_{\mathbf{y}}\epsilon_{0})}{1 - \tau} \qquad \forall 0 \leq k \in \mathbb{K} - 1.$$
(41)

Proof. One can observe from (18), (20) and Algorithm 1 that $F_{\text{hi}} \ge f_{\text{low}}^*$ and $\rho_0 \ge 1 > \tau > 0$, which imply that (41) holds for k = 0. It remains to show that (41) holds for all $1 \le k \in \mathbb{K} - 1$.

Since (x^{t+1}, y^{t+1}) is an ϵ_t -stationary point of (9) for all $0 \leq t \in \mathbb{K} - 1$, it follows from Definition 1 that there exists some $u \in \partial_y \mathcal{L}(x^{t+1}, y^{t+1}, \lambda_{\mathbf{x}}^t, \lambda_{\mathbf{y}}^t; \rho_t, \rho_t)$ with $||u|| \leq \epsilon_t$. Notice from (3) and (36) that $\partial_y \mathcal{L}(x^{t+1}, y^{t+1}, \lambda_{\mathbf{x}}^t, \lambda_{\mathbf{y}}^t; \rho_t, \rho_t) = \partial_y \mathcal{L}_{\mathbf{y}}(x^{t+1}, y^{t+1}, \lambda_{\mathbf{y}}^t; \rho_t)$. Hence, $u \in$ $\partial_y \mathcal{L}_{\mathbf{y}}(x^{t+1}, y^{t+1}, \lambda_{\mathbf{y}}^t; \rho_t)$. Also, observe from (1), (36) and Assumption 1 that $\mathcal{L}_{\mathbf{y}}(x^{t+1}, \cdot, \lambda_{\mathbf{y}}^t; \rho_t)$ is concave. Using this, (19), $u \in \partial_y \mathcal{L}_{\mathbf{y}}(x^{t+1}, y^{t+1}, \lambda_{\mathbf{y}}^t; \rho_t)$ and $||u|| \leq \epsilon_t$, we obtain

$$\begin{aligned} \mathcal{L}_{\mathbf{y}}(x^{t+1}, y, \lambda_{\mathbf{y}}^{t}; \rho_{t}) &\leq \mathcal{L}_{\mathbf{y}}(x^{t+1}, y^{t+1}, \lambda_{\mathbf{y}}^{t}; \rho_{t}) + \langle u, y - y^{t+1} \rangle \\ &\leq \mathcal{L}_{\mathbf{y}}(x^{t+1}, y^{t+1}, \lambda_{\mathbf{y}}^{t}; \rho_{t}) + D_{\mathbf{y}} \epsilon_{t} \qquad \forall y \in \mathcal{Y}, \end{aligned}$$

which implies that

$$\max_{y} \mathcal{L}_{\mathbf{y}}(x^{t+1}, y, \lambda_{\mathbf{y}}^{t}; \rho_{t}) \leq \mathcal{L}_{\mathbf{y}}(x^{t+1}, y^{t+1}, \lambda_{\mathbf{y}}^{t}; \rho_{t}) + D_{\mathbf{y}}\epsilon_{t}.$$
(42)

By this, (36) and (37), one has

$$f^{*}(x^{t+1}) \stackrel{(37)}{\leq} \max_{y} \mathcal{L}_{\mathbf{y}}(x^{t+1}, y, \lambda_{\mathbf{y}}^{t}; \rho_{t})$$

$$\stackrel{(36)(42)}{\leq} F(x^{t+1}, y^{t+1}) - \frac{1}{2\rho_{t}} \left(\| [\lambda_{\mathbf{y}}^{t} + \rho_{t}d(x^{t+1}, y^{t+1})]_{+} \|^{2} - \|\lambda_{\mathbf{y}}^{t}\|^{2} \right) + D_{\mathbf{y}}\epsilon_{t}$$

$$= F(x^{t+1}, y^{t+1}) - \frac{1}{2\rho_{t}} \left(\| \lambda_{\mathbf{y}}^{t+1} \|^{2} - \| \lambda_{\mathbf{y}}^{t} \|^{2} \right) + D_{\mathbf{y}}\epsilon_{t},$$

where the equality follows from the relation $\lambda_{\mathbf{y}}^{t+1} = [\lambda_{\mathbf{y}}^t + \rho_t d(x^{t+1}, y^{t+1})]_+$ (see Algorithm 1). Using the above inequality, (18), (20) and $\epsilon_t \leq \epsilon_0$ (see Algorithm 1), we have

$$\|\lambda_{\mathbf{y}}^{t+1}\|^2 - \|\lambda_{\mathbf{y}}^t\|^2 \le 2\rho_k(F(x^{t+1}, y^{t+1}) - f^*(x^{t+1}) + D_{\mathbf{y}}\epsilon_t) \le 2\rho_t(F_{\mathrm{hi}} - f_{\mathrm{low}}^* + D_{\mathbf{y}}\epsilon_0).$$

Summing up this inequality for t = 0, ..., k - 1 with $1 \le k \in \mathbb{K} - 1$ yields

$$\|\lambda_{\mathbf{y}}^{k}\|^{2} \leq \|\lambda_{\mathbf{y}}^{0}\|^{2} + 2(F_{\text{hi}} - f_{\text{low}}^{*} + D_{\mathbf{y}}\epsilon_{0})\sum_{t=0}^{k-1} \rho_{t}.$$
(43)

Recall from Algorithm 1 that $\rho_t = \epsilon_t^{-1} = (\epsilon_0 \tau^t)^{-1}$. Then we have $\sum_{t=0}^{k-1} \rho_t \leq \rho_{k-1}/(1-\tau)$. Using this, (43) and $\rho_k > \rho_{k-1} \geq 1$ (see Algorithm 1), we obtain that for all $1 \leq k \in \mathbb{K} - 1$,

$$\rho_k^{-1} \|\lambda_{\mathbf{y}}^k\|^2 \le \rho_k^{-1} \left(\|\lambda_{\mathbf{y}}^0\|^2 + \frac{2(F_{\text{hi}} - f_{\text{low}}^* + D_{\mathbf{y}}\epsilon_0)\rho_{k-1}}{1 - \tau} \right) \le \|\lambda_{\mathbf{y}}^0\|^2 + \frac{2(F_{\text{hi}} - f_{\text{low}}^* + D_{\mathbf{y}}\epsilon_0)}{1 - \tau}.$$

Hence, the conclusion holds as desired.

Lemma 3. Suppose that Assumptions 1 and 3 hold. Let f_{low}^* , $D_{\mathbf{y}}$ and F_{hi} be defined in (18), (19) and (20), L_F and δ_d be given in Assumptions 1 and 3, and ϵ_0 , τ , ϵ_k and ρ_k be given in Algorithm 1. Suppose that $(x^{k+1}, y^{k+1}, \lambda_{\mathbf{y}}^{k+1})$ is generated by Algorithm 1 for some $0 \leq k \in \mathbb{K}-1$ with

$$\rho_k \ge \frac{4\|\lambda_{\mathbf{y}}^0\|^2}{\delta_d^2} + \frac{8(F_{\rm hi} - f_{\rm low}^* + D_{\mathbf{y}}\epsilon_0)}{\delta_d^2(1 - \tau)}.$$
(44)

Then we have

$$\|[d(x^{k+1}, y^{k+1})]_{+}\| \le \rho_{k}^{-1} \|\lambda_{\mathbf{y}}^{k+1}\| \le 2\rho_{k}^{-1}\delta_{d}^{-1}(\epsilon_{0} + L_{F})D_{\mathbf{y}}.$$
(45)

Proof. Suppose that $(x^{k+1}, y^{k+1}, \lambda_{\mathbf{y}}^{k+1})$ is generated by Algorithm 1 for some $0 \leq k \in \mathbb{K} - 1$ with ρ_k satisfying (44). Since (x^{k+1}, y^{k+1}) is an ϵ_k -stationary point of (9), it follows from (3) and Definition 1 that

dist
$$\left(0, \partial_y F(x^{k+1}, y^{k+1}) - \nabla_y d(x^{k+1}, y^{k+1}) [\lambda_{\mathbf{y}}^k + \rho_k d(x^{k+1}, y^{k+1})]_+\right) \le \epsilon_k.$$

Besides, notice from Algorithm 1 that $\lambda_{\mathbf{y}}^{k+1} = [\lambda_{\mathbf{y}}^k + \rho_k d(x^{k+1}, y^{k+1})]_+$. Hence, there exists some $u \in \partial_y F(x^{k+1}, y^{k+1})$ such that

$$\|u - \nabla_y d(x^{k+1}, y^{k+1})\lambda_{\mathbf{y}}^{k+1}\| \le \epsilon_k.$$

$$\tag{46}$$

By Assumption 3(ii), there exists some $\hat{y}^{k+1} \in \mathcal{Y}$ such that $-d_i(x^{k+1}, \hat{y}^{k+1}) \geq \delta_d$ for all i. Notice that $\langle \lambda_{\mathbf{y}}^{k+1}, \lambda_{\mathbf{y}}^k + \rho_k d(x^{k+1}, y^{k+1}) \rangle = \|[\lambda_{\mathbf{y}}^k + \rho_k d(x^{k+1}, y^{k+1})]_+\|^2 \geq 0$, which implies that

$$-\langle \lambda_{\mathbf{y}}^{k+1}, \rho_k^{-1} \lambda_{\mathbf{y}}^k \rangle \le \langle \lambda_{\mathbf{y}}^{k+1}, d(x^{k+1}, y^{k+1}) \rangle.$$
(47)

Using these and (46), we have

$$F(x^{k+1}, \hat{y}^{k+1}) - F(x^{k+1}, y^{k+1}) + \delta_d \|\lambda_{\mathbf{y}}^{k+1}\|_1 - \rho_k^{-1} \langle \lambda_{\mathbf{y}}^{k+1}, \lambda_{\mathbf{y}}^k \rangle$$

$$\leq F(x^{k+1}, \hat{y}^{k+1}) - F(x^{k+1}, y^{k+1}) - \langle \lambda_{\mathbf{y}}^{k+1}, \rho_k^{-1} \lambda_{\mathbf{y}}^k + d(x^{k+1}, \hat{y}^{k+1}) \rangle$$

$$\stackrel{(47)}{\leq} F(x^{k+1}, \hat{y}^{k+1}) - F(x^{k+1}, y^{k+1}) + \langle \lambda_{\mathbf{y}}^{k+1}, d(x^{k+1}, y^{k+1}) - d(x^{k+1}, \hat{y}^{k+1})) \rangle$$

$$\leq \langle u, \hat{y}^{k+1} - y^{k+1} \rangle + \langle \nabla_y d(x^{k+1}, y^{k+1}) \lambda_{\mathbf{y}}^{k+1}, y^{k+1} - \hat{y}^{k+1} \rangle$$

$$= \langle u - \nabla_y d(x^{k+1}, y^{k+1}) \lambda_{\mathbf{y}}^{k+1}, y^{k+1} - \hat{y}^{k+1} \rangle \leq D_{\mathbf{y}} \epsilon_k, \qquad (48)$$

where the first inequality is due to $\lambda_{\mathbf{y}}^{k+1} \geq 0$ and $-d_i(x^{k+1}, \hat{y}^{k+1}) \geq \delta_d$ for all *i*, the third inequality follows from $u \in \partial_y F(x^{k+1}, y^{k+1}), \lambda_{\mathbf{y}}^{k+1} \geq 0$, the concavity of $F(x^{k+1}, \cdot)$ and the convexity of $d_i(x^{k+1}, \cdot)$, and the last inequality is due to (19) and (46).

In view of (19), (48) and the Lipschitz continuity of F (see Assumption 1), one has

$$D_{\mathbf{y}}\epsilon_{k} + L_{F}D_{\mathbf{y}} \stackrel{(19)}{\geq} D_{\mathbf{y}}\epsilon_{k} + L_{F} \|\hat{y}^{k+1} - y^{k+1}\| \geq D_{\mathbf{y}}\epsilon_{k} - F(x^{k+1}, \hat{y}^{k+1}) + F(x^{k+1}, y^{k+1}) \\ \stackrel{(48)}{\geq} \delta_{d} \|\lambda_{\mathbf{y}}^{k+1}\|_{1} - \rho_{k}^{-1} \langle\lambda_{\mathbf{y}}^{k+1}, \lambda_{\mathbf{y}}^{k}\rangle \geq (\delta_{d} - \rho_{k}^{-1} \|\lambda_{\mathbf{y}}^{k}\|) \|\lambda_{\mathbf{y}}^{k+1}\|,$$
(49)

where the second inequality follows from L_F -Lipschitz continuity of F, and the last inequality is due to $\|\lambda_{\mathbf{y}}^{k+1}\|_1 \ge \|\lambda_{\mathbf{y}}^{k+1}\|$. In addition, it follows from (41) and (44) that

$$\delta_d - \rho_k^{-1} \|\lambda_{\mathbf{y}}^k\| \stackrel{(41)}{\geq} \delta_d - \sqrt{\rho_k^{-1} \left(\|\lambda_{\mathbf{y}}^0\|^2 + \frac{2(F_{\mathrm{hi}} - f_{\mathrm{low}}^* + D_{\mathbf{y}}\epsilon_0)}{1 - \tau} \right)} \stackrel{(44)}{\geq} \frac{1}{2} \delta_d,$$

which together with (49) yields

$$\frac{1}{2}\delta_d \|\lambda_{\mathbf{y}}^{k+1}\| \le (\delta_d - \rho_k^{-1} \|\lambda_{\mathbf{y}}^k\|) \|\lambda_{\mathbf{y}}^{k+1}\| \stackrel{(49)}{\le} D_{\mathbf{y}}\epsilon_k + L_F D_{\mathbf{y}}.$$

The conclusion then follows from this, $\epsilon_k \leq \epsilon_0$, and the relations

$$\|[d(x^{k+1}, y^{k+1})]_+\| \le \rho_k^{-1} \|[\lambda_{\mathbf{y}}^k + \rho_k d(x^{k+1}, y^{k+1})]_+\| = \rho_k^{-1} \|\lambda_{\mathbf{y}}^{k+1}\|.$$

Lemma 4. Suppose that Assumptions 1 and 3 hold. Let f_{low}^* , $D_{\mathbf{y}}$ and F_{low} be defined in (18), (19) and (20), L_F and δ_d be given in Assumptions 1 and 3, ϵ_0 , τ , ϵ_k , ρ_k and $\lambda_{\mathbf{y}}^0$ be given in Algorithm 1. Suppose that $(x^{k+1}, y^{k+1}, \lambda_{\mathbf{x}}^{k+1}, \lambda_{\mathbf{y}}^{k+1})$ is generated by Algorithm 1 for some $0 \leq k \in \mathbb{K} - 1$ with

$$\rho_k \ge \frac{4\|\lambda_{\mathbf{y}}^0\|^2}{\delta_d^2 \tau} + \frac{8(F_{\rm hi} - f_{\rm low}^* + D_{\mathbf{y}}\epsilon_0)}{\delta_d^2 \tau (1 - \tau)}.$$
(50)

Let

$$\tilde{\lambda}_{\mathbf{x}}^{k+1} = [\lambda_{\mathbf{x}}^k + \rho_k c(x^{k+1})]_+.$$
(51)

Then we have

$$dist(0, \partial_x F(x^{k+1}, y^{k+1}) + \nabla c(x^{k+1})\tilde{\lambda}_{\mathbf{x}}^{k+1} - \nabla_x d(x^{k+1}, y^{k+1})\lambda_{\mathbf{y}}^{k+1}) \le \epsilon_k,$$
(52)

$$\operatorname{dist}\left(0, \partial_{y}F(x^{k+1}, y^{k+1}) - \nabla_{y}d(x^{k+1}, y^{k+1})\lambda_{\mathbf{y}}^{k+1}\right) \leq \epsilon_{k},\tag{53}$$

$$\|[d(x^{k+1}, y^{k+1})]_+\| \le 2\rho_k^{-1}\delta_d^{-1}(\epsilon_0 + L_F)D_{\mathbf{y}},\tag{54}$$

$$|\langle \lambda_{\mathbf{y}}^{k+1}, d(x^{k+1}, y^{k+1}) \rangle| \le 2\rho_k^{-1} \delta_d^{-1}(\epsilon_0 + L_F) D_{\mathbf{y}} \max\{\|\lambda_{\mathbf{y}}^0\|, \ 2\delta_d^{-1}(\epsilon_0 + L_F) D_{\mathbf{y}}\}.$$
 (55)

Proof. Suppose that $(x^{k+1}, y^{k+1}, \lambda_{\mathbf{x}}^{k+1}, \lambda_{\mathbf{y}}^{k+1})$ is generated by Algorithm 1 for some $0 \leq k \in \mathbb{K}-1$ with ρ_k satisfying (50). Since (x^{k+1}, y^{k+1}) is an ϵ_k -stationary point of (9), it then follows from Definition 1 that

$$\operatorname{dist}(0, \partial_x \mathcal{L}(x^{k+1}, y^{k+1}, \lambda_{\mathbf{x}}^k, \lambda_{\mathbf{y}}^k; \rho_k)) \leq \epsilon_k, \ \operatorname{dist}(0, \partial_y \mathcal{L}(x^{k+1}, y^{k+1}, \lambda_{\mathbf{x}}^k, \lambda_{\mathbf{y}}^k; \rho_k)) \leq \epsilon_k.$$
(56)

Observe from Algorithm 1 that $\lambda_{\mathbf{y}}^{k+1} = [\lambda_{\mathbf{y}}^k + \rho_k d(x^{k+1}, y^{k+1})]_+$. In view of this, (3) and (51), one has

$$\begin{split} \partial_x \mathcal{L}(x^{k+1}, y^{k+1}, \lambda_{\mathbf{x}}^k, \lambda_{\mathbf{y}}^k; \rho_k) &= \partial_x F(x^{k+1}, y^{k+1}) + \nabla c(x^{k+1}) [\lambda_{\mathbf{x}}^k + \rho_k c(x^{k+1})]_+ \\ &\quad - \nabla_x d(x^{k+1}, y^{k+1}) [\lambda_{\mathbf{y}}^k + \rho_k d(x^{k+1}, y^{k+1})]_+ \\ &= \partial_x F(x^{k+1}, y^{k+1}) + \nabla c(x^{k+1}) \tilde{\lambda}_{\mathbf{x}}^{k+1} - \nabla_x d(x^{k+1}, y^{k+1}) \lambda_{\mathbf{y}}^{k+1}, \\ \partial_y \mathcal{L}(x^{k+1}, y^{k+1}, \lambda_{\mathbf{x}}^k, \lambda_{\mathbf{y}}^k; \rho_k) &= \partial_y F(x^{k+1}, y^{k+1}) - \nabla_y d(x^{k+1}, y^{k+1}) \lambda_{\mathbf{y}}^{k+1}. \end{split}$$

These relations together with (56) imply that (52) and (53) hold.

Notice from Algorithm 1 that $0 < \tau < 1$, which together with (50) implies that (44) holds for ρ_k . It then follows that (45) holds, which immediately yields (54) and

$$\|\lambda_{\mathbf{y}}^{k+1}\| \le 2\delta_d^{-1}(\epsilon_0 + L_F)D_{\mathbf{y}}.$$
(57)

Claim that

$$|\lambda_{\mathbf{y}}^{k}| \leq \max\{\|\lambda_{\mathbf{y}}^{0}\|, \ 2\delta_{d}^{-1}(\epsilon_{0} + L_{F})D_{\mathbf{y}}\}.$$
(58)

Indeed, (58) clearly holds if k = 0. We now assume that k > 0. Notice from Algorithm 1 that $\rho_{k-1} = \tau \rho_k$, which together with (50) implies that (44) holds with k replaced by k - 1. By this and Lemma 3 with k replaced by k - 1, one can conclude that $\|\lambda_{\mathbf{y}}^k\| \leq 2\delta_d^{-1}(\epsilon_0 + L_F)D_{\mathbf{y}}$ and hence (58) holds.

We next show that (55) holds. Indeed, by $\lambda_{\mathbf{y}}^{k+1} \ge 0$, (47), (54), (57) and (58), one has

$$\langle \lambda_{\mathbf{y}}^{k+1}, d(x^{k+1}, y^{k+1}) \rangle \leq \langle \lambda_{\mathbf{y}}^{k+1}, [d(x^{k+1}, y^{k+1})]_{+} \rangle \leq \|\lambda_{\mathbf{y}}^{k+1}\| \| [d(x^{k+1}, y^{k+1})]_{+} \|$$

$$\stackrel{(54)(57)}{\leq} 4\rho_{k}^{-1}\delta_{d}^{-2}(\epsilon_{0} + L_{F})^{2}D_{\mathbf{y}}^{2},$$

$$\langle \lambda_{\mathbf{y}}^{k+1}, d(x^{k+1}, y^{k+1}) \rangle \stackrel{(47)}{\geq} \langle \lambda_{\mathbf{y}}^{k+1}, -\rho_{k}^{-1}\lambda_{\mathbf{y}}^{k} \rangle \geq -\rho_{k}^{-1} \|\lambda_{\mathbf{y}}^{k+1}\| \| \lambda_{\mathbf{y}}^{k} \|$$

$$\geq -2\rho_{k}^{-1}\delta_{d}^{-1}(\epsilon_{0} + L_{F})D_{\mathbf{y}} \max\{\|\lambda_{\mathbf{y}}^{0}\|, 2\delta_{d}^{-1}(\epsilon_{0} + L_{F})D_{\mathbf{y}}\}.$$

These relations imply that (55) holds.

Lemma 5. Suppose that Assumptions 1, 2 and 3 hold. Let $\{(\lambda_{\mathbf{x}}^k, \lambda_{\mathbf{y}}^k)\}_{k \in \mathbb{K}}$ be generated by Algorithm 1, \mathcal{L} , f_{low}^* , $D_{\mathbf{y}}$ and F_{hi} be defined in (3), (18), (19) and (20), L_F be given in Assumption 1, and ϵ_0 , τ , ρ_k , Λ and x_{init}^k be given in Algorithm 1. Then for all $0 \leq k \in \mathbb{K} - 1$, we have

$$\max_{\mathbf{y}} \mathcal{L}(x_{\text{init}}^k, \mathbf{y}, \lambda_{\mathbf{x}}^k, \lambda_{\mathbf{y}}^k; \rho_k) \le L_F D_{\mathbf{y}} + F_{\text{hi}} + \Lambda + \frac{1}{2}(\tau^{-1} + \|\lambda_{\mathbf{y}}^0\|^2) + \frac{F_{\text{hi}} - f_{\text{low}}^* + D_{\mathbf{y}}\epsilon_0}{1 - \tau}.$$
 (59)

Proof. In view of (6), (8), (20) and $\|\lambda_{\mathbf{x}}^k\| \leq \Lambda$ (see Algorithm 1), one has

$$\mathcal{L}_{\mathbf{x}}(x_{\text{init}}^{k}, y^{k}, \lambda_{\mathbf{x}}^{k}; \rho_{k}) \stackrel{(8)}{\leq} \mathcal{L}_{\mathbf{x}}(x_{\mathbf{nf}}, y^{k}, \lambda_{\mathbf{x}}^{k}; \rho_{k}) \stackrel{(6)}{=} F(x_{\mathbf{nf}}, y^{k}) + \frac{1}{2\rho_{k}} \left(\|[\lambda_{\mathbf{x}}^{k} + \rho_{k}c(x_{\mathbf{nf}})]_{+}\|^{2} - \|\lambda_{\mathbf{x}}^{k}\|^{2} \right) \\ \leq F(x_{\mathbf{nf}}, y^{k}) + \frac{1}{2\rho_{k}} \left((\|\lambda_{\mathbf{x}}^{k}\| + \rho_{k}\|[c(x_{\mathbf{nf}})]_{+}\|)^{2} - \|\lambda_{\mathbf{x}}^{k}\|^{2} \right) \\ = F(x_{\mathbf{nf}}, y^{k}) + \|\lambda_{\mathbf{x}}^{k}\|\|[c(x_{\mathbf{nf}})]_{+}\| + \frac{1}{2}\rho_{k}\|[c(x_{\mathbf{nf}})]_{+}\|^{2} \\ \stackrel{(20)}{\leq} F_{\mathrm{hi}} + \Lambda\|[c(x_{\mathbf{nf}})]_{+}\| + \frac{1}{2}\rho_{k}\|[c(x_{\mathbf{nf}})]_{+}\|^{2}.$$

$$(60)$$

In addition, one can observe from Algorithm 1 that $\epsilon_k > \tau \varepsilon$ for all $0 \le k \in \mathbb{K} - 1$. By this and the choice of ρ_k in Algorithm 1, we obtain that $\rho_k = \epsilon_k^{-1} < \tau^{-1} \varepsilon^{-1}$ for all $0 \le k \in \mathbb{K} - 1$. It then follows from this, (3), (6), (19), (41), (60), $\|[c(x_{\mathbf{nf}})]_+\| \le \sqrt{\varepsilon} \le 1$, and the Lipschitz continuity

of F that

$$\begin{split} &\max_{y} \mathcal{L}(x_{\text{init}}^{k}, y, \lambda_{\mathbf{x}}^{k}, \lambda_{\mathbf{y}}^{k}; \rho_{k}) \stackrel{(3)(6)}{=} \max_{y} \left\{ \mathcal{L}_{\mathbf{x}}(x_{\text{init}}^{k}, y, \lambda_{\mathbf{x}}^{k}; \rho_{k}) - \frac{1}{2\rho_{k}} \left(\|[\lambda_{\mathbf{y}}^{k} + \rho_{k}d(x_{\text{init}}^{k}, y)]_{+}\|^{2} - \|\lambda_{\mathbf{y}}^{k}\|^{2} \right) \right\} \\ &\leq \max_{y} \left\{ \mathcal{L}_{\mathbf{x}}(x_{\text{init}}^{k}, y, \lambda_{\mathbf{x}}^{k}; \rho_{k}) + \frac{1}{2\rho_{k}} \|\lambda_{\mathbf{y}}^{k}\|^{2} \right\} \\ &\stackrel{(6)}{=} \max_{y} \left\{ F(x_{\text{init}}^{k}, y) - F(x_{\text{init}}^{k}, y^{k}) + \mathcal{L}_{\mathbf{x}}(x_{\text{init}}^{k}, y^{k}, \lambda_{\mathbf{x}}^{k}; \rho_{k}) + \frac{1}{2\rho_{k}} \|\lambda_{\mathbf{y}}^{k}\|^{2} \right\} \\ &\leq \max_{y \in \mathcal{Y}} L_{F} \|y - y^{k}\| + \mathcal{L}_{\mathbf{x}}(x_{\text{init}}^{k}, y^{k}, \lambda_{\mathbf{x}}^{k}; \rho_{k}) + \frac{1}{2\rho_{k}} \|\lambda_{\mathbf{y}}^{k}\|^{2} \\ &\leq L_{F} D_{\mathbf{y}} + F_{\text{hi}} + \Lambda \|[c(x_{\mathbf{nf}})]_{+}\| + \frac{1}{2}\rho_{k}\|[c(x_{\mathbf{nf}})]_{+}\|^{2} + \frac{1}{2}\|\lambda_{\mathbf{y}}^{0}\|^{2} + \frac{F_{\text{hi}} - f_{\text{low}}^{*} + D_{\mathbf{y}}\epsilon_{0}}{1 - \tau} \\ &\leq L_{F} D_{\mathbf{y}} + F_{\text{hi}} + \Lambda + \frac{1}{2}(\tau^{-1} + \|\lambda_{\mathbf{y}}^{0}\|^{2}) + \frac{F_{\text{hi}} - f_{\text{low}}^{*} + D_{\mathbf{y}}\epsilon_{0}}{1 - \tau}, \end{split}$$

where the second inequality follows from L_F -Lipschitz continuity of F (see Assumption 1(i)), the third inequality follows from (19), (41) and (60), and the last inequality follows from $\rho_k < \tau^{-1} \varepsilon^{-1}$ and $\|[c(x_{\mathbf{nf}})]_+\| \leq \sqrt{\varepsilon} \leq 1$.

Lemma 6. Suppose that Assumptions 1, 2 and 3 hold. Let L_k , f_{low}^* , D_x , D_y , F_{hi} and F_{low} be defined in (10), (18), (19) and (20), L_F be given in Assumption 1, ϵ_0 , τ , ϵ_k , ρ_k , Λ and λ_y^0 be given in Algorithm 1, and

$$\alpha_k = \min\left\{1, \sqrt{4\epsilon_k/(D_{\mathbf{y}}L_k)}\right\},\tag{61}$$

$$\delta_k = (2 + \alpha_k^{-1}) L_k D_{\mathbf{x}}^2 + \max\left\{\epsilon_k / D_{\mathbf{y}}, \alpha_k L_k / 4\right\} D_{\mathbf{y}}^2, \tag{62}$$

$$M_{k} = \frac{16 \max\left\{1/(2L_{k}), \min\left\{D_{\mathbf{y}}/\epsilon_{k}, 4/(\alpha_{k}L_{k})\right\}\right\}\rho_{k}}{\left[(3L_{k} + \epsilon_{k}/(2D_{\mathbf{y}}))^{2}/\min\{L_{k}, \epsilon_{k}/(2D_{\mathbf{y}})\} + 3L_{k} + \epsilon_{k}/(2D_{\mathbf{y}})\right]^{-2}\epsilon_{k}^{2}} \times \left(\delta_{k} + 2\alpha_{k}^{-1}\left(F_{\mathrm{hi}} - F_{\mathrm{low}}\right) + \frac{\Lambda^{2}}{2\rho_{k}} + \frac{3}{2}\|\lambda_{\mathbf{y}}^{0}\|^{2} + \frac{3(F_{\mathrm{hi}} - f_{\mathrm{low}}^{*} + D_{\mathbf{y}}\epsilon_{0})}{1 - \tau} + \rho_{k}d_{\mathrm{hi}}^{2} + \frac{\epsilon_{k}D_{\mathbf{y}}}{4} + L_{k}D_{\mathbf{x}}^{2}\right)\right)$$
(63)

$$T_{k} = \left[16 \left(L_{F} D_{\mathbf{y}} + F_{\mathrm{hi}} - f_{\mathrm{low}}^{*} + \Lambda + \frac{1}{2} (\tau^{-1} + \|\lambda_{\mathbf{y}}^{0}\|^{2}) + \frac{F_{\mathrm{hi}} - f_{\mathrm{low}}^{*} + D_{\mathbf{y}} \epsilon_{0}}{1 - \tau} + \frac{\Lambda^{2}}{2\rho_{k}} + \frac{\epsilon_{k} D_{\mathbf{y}}}{4} \right) L_{k} \epsilon_{k}^{-2} + 8(1 + 4D_{\mathbf{y}}^{2} L_{k}^{2} \epsilon_{k}^{-2}) \rho_{k}^{-1} - 1 \right]_{+},$$

$$(64)$$

$$N_{k} = \left(\left\lceil 96\sqrt{2} \left(1 + (24L_{k} + 4\epsilon_{k}/D_{\mathbf{y}}) L_{k}^{-1} \right) \right\rceil + 2 \right) \max \left\{ 2, \sqrt{D_{\mathbf{y}}L_{k}\epsilon_{k}^{-1}} \right\} \times \left((T_{k} + 1)(\log M_{k})_{+} + T_{k} + 1 + 2T_{k}\log(T_{k} + 1)) \right).$$
(65)

Then for all $0 \leq k \in \mathbb{K} - 1$, Algorithm 1 finds an ϵ_k -stationary point (x^{k+1}, y^{k+1}) of problem (9) that satisfies

$$\max_{y} \mathcal{L}(x^{k+1}, y, \lambda_{\mathbf{x}}^{k}, \lambda_{\mathbf{y}}^{k}; \rho_{k}) \leq L_{F} D_{\mathbf{y}} + F_{\mathrm{hi}} + \Lambda + \frac{1}{2} (\tau^{-1} + \|\lambda_{\mathbf{y}}^{0}\|^{2}) + \frac{F_{\mathrm{hi}} - f_{\mathrm{low}}^{*} + D_{\mathbf{y}} \epsilon_{0}}{1 - \tau} + \frac{\epsilon_{k} D_{\mathbf{y}}}{4} + \frac{1}{2\rho_{k}} \left(L_{k}^{-1} \epsilon_{k}^{2} + 4D_{\mathbf{y}}^{2} L_{k} \right).$$
(66)

Moreover, the total number of evaluations of ∇f , ∇c , ∇d and proximal operator of p and q performed in iteration k of Algorithm 1 is no more than N_k , respectively.

Proof. Observe from (1) and (3) that problem (9) can be viewed as

$$\min_{x} \max_{y} \{h(x,y) + p(x) - q(y)\}$$

where

$$h(x,y) = f(x,y) + \frac{1}{2\rho_k} \left(\|[\lambda_{\mathbf{x}}^k + \rho_k c(x)]_+\|^2 - \|\lambda_{\mathbf{x}}^k\|^2 \right) - \frac{1}{2\rho_k} \left(\|[\lambda_{\mathbf{y}}^k + \rho_k d(x,y)]_+\|^2 - \|\lambda_{\mathbf{y}}^k\|^2 \right).$$

Notice that

$$\nabla_x h(x,y) = \nabla_x f(x,y) + \nabla c(x) [\lambda_{\mathbf{x}}^k + \rho_k c(x)]_+ + \nabla_x d(x,y) [\lambda_{\mathbf{y}}^k + \rho_k d(x,y)]_+,$$

$$\nabla_y h(x,y) = \nabla_y f(x,y) + \nabla_y d(x,y) [\lambda_{\mathbf{y}}^k + \rho_k d(x,y)]_+.$$

It follows from Assumption 1(iii) that

$$\|\nabla c(x)\| \le L_c, \quad \|\nabla d(x,y)\| \le L_d \qquad \forall (x,y) \in \mathcal{X} \times \mathcal{Y}.$$

In view of the above relations, (7) and Assumption 1, one can observe that $\nabla c(x)[\lambda_{\mathbf{x}}^{k} + \rho_{k}c(x)]_{+}$ is $(\rho_{k}L_{c}^{2} + \rho_{k}c_{\mathrm{hi}}L_{\nabla c} + \|\lambda_{\mathbf{x}}^{k}\|L_{\nabla c})$ -Lipschitz continuous on \mathcal{X} , and $\nabla d(x,y)[\lambda_{\mathbf{y}}^{k} + \rho_{k}d(x,y)]_{+}$ is $(\rho_{k}L_{d}^{2} + \rho_{k}d_{\mathrm{hi}}L_{\nabla d} + \|\lambda_{\mathbf{y}}^{k}\|L_{\nabla d})$ -Lipschitz continuous on $\mathcal{X} \times \mathcal{Y}$. Using these and the fact that $\nabla f(x,y)$ is $L_{\nabla f}$ -Lipschitz continuous on $\mathcal{X} \times \mathcal{Y}$, we can see that h(x,y) is L_{k} -smooth on $\mathcal{X} \times \mathcal{Y}$ for all $0 \leq k \in \mathbb{K} - 1$, where L_{k} is given in (10). Consequently, it follows from Theorem 2 that Algorithm 3 can be suitably applied to problem (9) for finding an ϵ_{k} -stationary point (x^{k+1}, y^{k+1}) of it.

In addition, by (3), (18), (36), (37) and $\|\lambda_{\mathbf{x}}^k\| \leq \Lambda$ (see Algorithm 1), one has

$$\min_{x} \max_{y} \mathcal{L}(x, y, \lambda_{\mathbf{x}}^{k}, \lambda_{\mathbf{y}}^{k}; \rho_{k}) \stackrel{(3)(36)}{=} \min_{x} \max_{y} \left\{ \mathcal{L}_{\mathbf{y}}(x, y, \lambda_{\mathbf{y}}^{k}; \rho_{k}) + \frac{1}{2\rho_{k}} \left(\|[\lambda_{\mathbf{x}}^{k} + \rho_{k}c(x)]_{+}\|^{2} - \|\lambda_{\mathbf{x}}^{k}\|^{2} \right) \right\} \stackrel{(37)}{\geq} \min_{x} \left\{ f^{*}(x) + \frac{1}{2\rho_{k}} \left(\|[\lambda_{\mathbf{x}}^{k} + \rho_{k}c(x)]_{+}\|^{2} - \|\lambda_{\mathbf{x}}^{k}\|^{2} \right) \right\} \stackrel{(18)}{\geq} f^{*}_{\mathrm{low}} - \frac{1}{2\rho_{k}} \|\lambda_{\mathbf{x}}^{k}\|^{2} \ge f^{*}_{\mathrm{low}} - \frac{\Lambda^{2}}{2\rho_{k}}.$$
(67)

Let (x^*, y^*) be an optimal solution of (1). It then follows that $c(x^*) \leq 0$. Using this, (3), (20) and (41), we obtain that

$$\min_{x} \max_{y} \mathcal{L}(x, y, \lambda_{\mathbf{x}}^{k}, \lambda_{\mathbf{y}}^{k}; \rho_{k}) \leq \max_{y} \mathcal{L}(x^{*}, y, \lambda_{\mathbf{x}}^{k}, \lambda_{\mathbf{y}}^{k}; \rho_{k})$$

$$\stackrel{(3)}{=} \max_{y} \left\{ F(x^{*}, y) + \frac{1}{2\rho_{k}} \left(\| [\lambda_{\mathbf{x}}^{k} + \rho_{k}c(x^{*})]_{+} \|^{2} - \|\lambda_{\mathbf{x}}^{k}\|^{2} \right) - \frac{1}{2\rho_{k}} \left(\| [\lambda_{\mathbf{y}}^{k} + \rho_{k}d(x^{*}, y)]_{+} \|^{2} - \|\lambda_{\mathbf{y}}^{k}\|^{2} \right) \right\}$$

$$\leq \max_{y} \left\{ F(x^{*}, y) - \frac{1}{2\rho_{k}} \left(\| [\lambda_{\mathbf{y}}^{k} + \rho_{k}d(x^{*}, y)]_{+} \|^{2} - \|\lambda_{\mathbf{y}}^{k}\|^{2} \right) \right\}$$

$$\stackrel{(20)}{\leq} F_{\mathrm{hi}} + \frac{1}{2\rho_{k}} \|\lambda_{\mathbf{y}}^{k}\|^{2} \stackrel{(41)}{\leq} F_{\mathrm{hi}} + \frac{1}{2} \|\lambda_{\mathbf{y}}^{0}\|^{2} + \frac{F_{\mathrm{hi}} - f_{\mathrm{low}}^{*} + D_{\mathbf{y}}\epsilon_{0}}{1 - \tau},$$
(68)

where the second inequality is due to $c(x^*) \leq 0$. Moreover, it follows from this, (3), (7), (20), (41), $\lambda_{\mathbf{y}}^k \in \mathbb{R}^{\tilde{m}}_+$ and $\|\lambda_{\mathbf{x}}^k\| \leq \Lambda$ that

$$\min_{(x,y)\in\mathcal{X}\times\mathcal{Y}} \mathcal{L}(x,y,\lambda_{\mathbf{x}}^{k},\lambda_{\mathbf{y}}^{k};\rho_{k}) \stackrel{(3)}{\geq} \min_{(x,y)\in\mathcal{X}\times\mathcal{Y}} \left\{ F(x,y) - \frac{1}{2\rho_{k}} \|\lambda_{\mathbf{x}}^{k}\|^{2} - \frac{1}{2\rho_{k}} \|[\lambda_{\mathbf{y}}^{k} + \rho_{k}d(x,y)]_{+}\|^{2} \right\}$$

$$\geq \min_{(x,y)\in\mathcal{X}\times\mathcal{Y}} \left\{ F(x,y) - \frac{1}{2\rho_{k}} \|\lambda_{\mathbf{x}}^{k}\|^{2} - \frac{1}{2\rho_{k}} \left(\|\lambda_{\mathbf{y}}^{k}\| + \rho_{k}\|[d(x,y)]_{+}\| \right)^{2} \right\}$$

$$\geq \min_{(x,y)\in\mathcal{X}\times\mathcal{Y}} \left\{ F(x,y) - \frac{1}{2\rho_{k}} \|\lambda_{\mathbf{x}}^{k}\|^{2} - \rho_{k}^{-1} \|\lambda_{\mathbf{y}}^{k}\|^{2} - \rho_{k}\|[d(x,y)]_{+}\|^{2} \right\}$$

$$\geq F_{\text{low}} - \frac{\Lambda^{2}}{2\rho_{k}} - \|\lambda_{\mathbf{y}}^{0}\|^{2} - \frac{2(F_{\text{hi}} - f_{\text{low}}^{*} + D_{\mathbf{y}}\epsilon_{0})}{1 - \tau} - \rho_{k}d_{\text{hi}}^{2},$$
(69)

where the second inequality is due to $\lambda_{\mathbf{y}}^k \in \mathbb{R}^{\hat{m}}_+$ and the last inequality is due to (7), (20), (41) and $\|\lambda_{\mathbf{x}}^k\| \leq \Lambda$.

To complete the rest of the proof, let

$$H(x,y) = \mathcal{L}(x,y,\lambda_{\mathbf{x}}^{k},\lambda_{\mathbf{y}}^{k};\rho_{k}), \quad H^{*} = \min_{x} \max_{y} \mathcal{L}(x,y,\lambda_{\mathbf{x}}^{k},\lambda_{\mathbf{y}}^{k};\rho_{k}), \tag{70}$$

$$H_{\text{low}} = \min_{(x,y)\in\mathcal{X}\times\mathcal{Y}} \mathcal{L}(x,y,\lambda_{\mathbf{x}}^{k},\lambda_{\mathbf{y}}^{k};\rho_{k}).$$
(71)

In view of these, (59), (67), (68), (69), we obtain that

$$\max_{\mathbf{y}} H(x_{\text{init}}^{k}, y) \stackrel{(59)}{\leq} L_{F} D_{\mathbf{y}} + F_{\text{hi}} + \Lambda + \frac{1}{2} (\tau^{-1} + \|\lambda_{\mathbf{y}}^{0}\|^{2}) + \frac{F_{\text{hi}} - f_{\text{low}}^{*} + D_{\mathbf{y}} \epsilon_{0}}{1 - \tau},$$

$$f_{\text{low}}^{*} - \frac{\Lambda^{2}}{2\rho_{k}} \stackrel{(67)}{\leq} H^{*} \stackrel{(68)}{\leq} F_{\text{hi}} + \frac{1}{2} \|\lambda_{\mathbf{y}}^{0}\|^{2} + \frac{F_{\text{hi}} - f_{\text{low}}^{*} + D_{\mathbf{y}} \epsilon_{0}}{1 - \tau},$$

$$H_{\text{low}} \stackrel{(69)}{\geq} F_{\text{low}} - \frac{\Lambda^{2}}{2\rho_{k}} - \|\lambda_{\mathbf{y}}^{0}\|^{2} - \frac{2(F_{\text{hi}} - f_{\text{low}}^{*} + D_{\mathbf{y}} \epsilon_{0})}{1 - \tau} - \rho_{k} d_{\text{hi}}^{2}.$$

Using these and Theorem 2 (see Appendix A) with $x^0 = x_{init}^k$, $D_p = D_x$, $D_q = D_y$, $\epsilon = \epsilon_k$, $\epsilon_0 = \epsilon_k/(2\sqrt{\rho_k})$, $L_{\nabla h} = L_k$, $\alpha = \alpha_k$, $\delta = \delta_k$, and H, H^* , H_{low} given in (70) and (71), we can conclude that Algorithm 3 performs at most N_k evaluations of ∇f , ∇c , ∇d and proximal operator of p and q for finding an ϵ_k -stationary point of problem (9) satisfying (66).

Lemma 7. Suppose that Assumptions 1, 2 and 3 hold. Let f_{low}^* , $D_{\mathbf{y}}$, F_{hi} and \hat{L} be defined in (18), (19), (20) and (23), L_F , L_c , δ_c , θ_f and θ_a be given in Assumptions 1 and 3, and ϵ_0 , τ , ρ_k , Λ and $\lambda_{\mathbf{y}}^0$ be given in Algorithm 1. Suppose that $(x^{k+1}, \lambda_{\mathbf{x}}^{k+1})$ is generated by Algorithm 1 for some $0 \leq k \in \mathbb{K} - 1$ with

$$\rho_{k} \geq \max\left\{\theta_{a}^{-1}\Lambda, \theta_{f}^{-2}\left\{2L_{F}D_{\mathbf{y}} + 2F_{\mathrm{hi}} - 2f_{\mathrm{low}}^{*} + 2\Lambda + \tau^{-1} + \|\lambda_{\mathbf{y}}^{0}\|^{2} + \frac{2(F_{\mathrm{hi}} - f_{\mathrm{low}}^{*} + D_{\mathbf{y}}\epsilon_{0})}{1 - \tau} + \frac{\epsilon_{0}D_{\mathbf{y}}}{2} + L_{c}^{-2} + 4D_{\mathbf{y}}^{2}\widehat{L} + \Lambda^{2}\right\}, \frac{4\|\lambda_{\mathbf{y}}^{0}\|^{2}}{\delta_{d}^{2}\tau} + \frac{8(F_{\mathrm{hi}} - f_{\mathrm{low}}^{*} + D_{\mathbf{y}}\epsilon_{0})}{\delta_{d}^{2}\tau(1 - \tau)}\right\}.$$
(72)

Let

$$\tilde{\lambda}_{\mathbf{x}}^{k+1} = [\lambda_{\mathbf{x}}^{k} + \rho_k c(x^{k+1})]_+.$$
(73)

Then we have

$$\|[c(x^{k+1})]_{+}\| \leq \rho_{k}^{-1} \delta_{c}^{-1} \left(L_{F} + 2L_{d} \delta_{d}^{-1}(\epsilon_{0} + L_{F}) D_{\mathbf{y}} + \epsilon_{0} \right),$$

$$|\langle \tilde{\lambda}_{\mathbf{x}}^{k+1}, c(x^{k+1}) \rangle| \leq \rho_{k}^{-1} \delta_{c}^{-1} (L_{F} + 2L_{d} \delta_{d}^{-1}(\epsilon_{0} + L_{F}) D_{\mathbf{y}} + \epsilon_{0}) \max\{\delta_{c}^{-1} (L_{F} + 2L_{d} \delta_{d}^{-1}(\epsilon_{0} + L_{F}) D_{\mathbf{y}} + \epsilon_{0}), \Lambda\}.$$
(74)
(74)
(75)

Proof. One can observe from (3), (18), (36) and (37) that

$$\max_{y} \mathcal{L}(x^{k+1}, y, \lambda_{\mathbf{x}}^{k}, \lambda_{\mathbf{y}}^{k}; \rho_{k}) = \max_{y} \mathcal{L}_{\mathbf{y}}(x^{k+1}, y, \lambda_{\mathbf{y}}^{k}; \rho_{k}) + \frac{1}{2\rho_{k}} \left(\|[\lambda_{\mathbf{x}}^{k} + \rho_{k}c(x^{k+1})]_{+}\|^{2} - \|\lambda_{\mathbf{x}}^{k}\|^{2} \right)$$

$$\stackrel{(37)}{\geq} f^{*}(x^{k+1}) + \frac{1}{2\rho_{k}} \left(\|[\lambda_{\mathbf{x}}^{k} + \rho_{k}c(x^{k+1})]_{+}\|^{2} - \|\lambda_{\mathbf{x}}^{k}\|^{2} \right)$$

$$\stackrel{(18)}{\geq} f^{*}_{\text{low}} + \frac{1}{2\rho_{k}} \left(\|[\lambda_{\mathbf{x}}^{k} + \rho_{k}c(x^{k+1})]_{+}\|^{2} - \|\lambda_{\mathbf{x}}^{k}\|^{2} \right).$$

By this inequality, (66) and $\|\lambda_{\mathbf{x}}^k\| \leq \Lambda$, one has

$$\begin{split} \| [\lambda_{\mathbf{x}}^{k} + \rho_{k}c(x^{k+1})]_{+} \|^{2} &\leq 2\rho_{k} \max_{\mathbf{y}} \mathcal{L}(x^{k+1}, y, \lambda_{\mathbf{x}}^{k}, \lambda_{\mathbf{y}}^{k}; \rho_{k}) - 2\rho_{k} f_{\text{low}}^{*} + \|\lambda_{\mathbf{x}}^{k}\|^{2} \\ &\leq 2\rho_{k} \max_{\mathbf{y}} \mathcal{L}(x^{k+1}, y, \lambda_{\mathbf{x}}^{k}, \lambda_{\mathbf{y}}^{k}; \rho_{k}) - 2\rho_{k} f_{\text{low}}^{*} + \Lambda^{2} \\ \overset{(66)}{\leq} 2\rho_{k} L_{F} D_{\mathbf{y}} + 2\rho_{k} F_{\text{hi}} + 2\rho_{k} \Lambda + \rho_{k} (\tau^{-1} + \|\lambda_{\mathbf{y}}^{0}\|^{2}) + \frac{2\rho_{k} (F_{\text{hi}} - f_{\text{low}}^{*} + D_{\mathbf{y}} \epsilon_{0})}{1 - \tau} + \frac{\rho_{k} \epsilon_{k} D_{\mathbf{y}}}{2} \\ &+ L_{k}^{-1} \epsilon_{k}^{2} + 4D_{\mathbf{y}}^{2} L_{k} - 2\rho_{k} f_{\text{low}}^{*} + \Lambda^{2}. \end{split}$$

This together with $\rho_k^2 \| [c(x^{k+1})]_+ \|^2 \le \| [\lambda_{\mathbf{x}}^k + \rho_k c(x^{k+1})]_+ \|^2$ implies that

$$\|[c(x^{k+1})]_{+}\|^{2} \leq \rho_{k}^{-1} \left(2L_{F}D_{\mathbf{y}} + 2F_{\mathrm{hi}} - 2f_{\mathrm{low}}^{*} + 2\Lambda + \tau^{-1} + \|\lambda_{\mathbf{y}}^{0}\|^{2} + \frac{2(F_{\mathrm{hi}} - f_{\mathrm{low}}^{*} + D_{\mathbf{y}}\epsilon_{0})}{1 - \tau} + \frac{\epsilon_{k}D_{\mathbf{y}}}{2} \right) + \rho_{k}^{-2} \left(L_{k}^{-1}\epsilon_{k}^{2} + 4D_{\mathbf{y}}^{2}L_{k} + \Lambda^{2} \right).$$

$$(76)$$

In addition, we observe from (10), (23), (41), $\rho_k \ge 1$ and $\|\lambda_{\mathbf{x}}^k\| \le \Lambda$ that for all $0 \le k \le K$,

$$\rho_{k}L_{c}^{2} \leq L_{k} = L_{\nabla f} + \rho_{k}L_{c}^{2} + \rho_{k}c_{\mathrm{hi}}L_{\nabla c} + \|\lambda_{\mathbf{x}}^{k}\|L_{\nabla c} + \rho_{k}L_{d}^{2} + \rho_{k}d_{\mathrm{hi}}L_{\nabla d} + \|\lambda_{\mathbf{y}}^{k}\|L_{\nabla d}$$

$$\leq L_{\nabla f} + \rho_{k}L_{c}^{2} + \rho_{k}c_{\mathrm{hi}}L_{\nabla c} + \Lambda L_{\nabla c} + \rho_{k}L_{d}^{2} + \rho_{k}d_{\mathrm{hi}}L_{\nabla d}$$

$$+ L_{\nabla d}\sqrt{\rho_{k}\left(\|\lambda_{\mathbf{y}}^{0}\|^{2} + \frac{2(F_{\mathrm{hi}} - f_{\mathrm{low}}^{*} + D_{y}\epsilon_{0})}{1 - \tau}\right)} \leq \rho_{k}\widehat{L}.$$
(77)

Using this relation, (72), (76), $\rho_k \ge 1$ and $\epsilon_k \le \epsilon_0$, we have

$$\begin{aligned} \|[c(x^{k+1})]_{+}\|^{2} &\leq \rho_{k}^{-1} \left(2L_{F}D_{\mathbf{y}} + 2F_{\mathrm{hi}} - f_{\mathrm{low}}^{*} + 2\Lambda + \tau^{-1} + \|\lambda_{\mathbf{y}}^{0}\|^{2} + \frac{2(F_{\mathrm{hi}} - f_{\mathrm{low}}^{*} + D_{\mathbf{y}}\epsilon_{0})}{1 - \tau} + \frac{\epsilon_{k}D_{\mathbf{y}}}{2} \right) \\ &+ \rho_{k}^{-2} \left((\rho_{k}L_{c}^{2})^{-1}\epsilon_{k}^{2} + 4\rho_{k}D_{\mathbf{y}}^{2}\widehat{L} + \Lambda^{2} \right) \\ &\leq \rho_{k}^{-1} \left(2L_{F}D_{\mathbf{y}} + 2F_{\mathrm{hi}} - f_{\mathrm{low}}^{*} + 2\Lambda + \tau^{-1} + \|\lambda_{\mathbf{y}}^{0}\|^{2} + \frac{2(F_{\mathrm{hi}} - f_{\mathrm{low}}^{*} + D_{\mathbf{y}}\epsilon_{0})}{1 - \tau} + \frac{\epsilon_{0}D_{\mathbf{y}}}{2} \right) \\ &+ \rho_{k}^{-1} \left(L_{c}^{-2} + 4D_{\mathbf{y}}^{2}\widehat{L} + \Lambda^{2} \right) \stackrel{(72)}{\leq} \theta_{f}^{2}, \end{aligned}$$

which together with (11) implies that $x^{k+1} \in \mathcal{F}(\theta_f)$.

It follows from $x^{k+1} \in \mathcal{F}(\theta_f)$ and Assumption 3(i) that there exists some v_x such that It follows from $x \in \mathcal{F}(b_f)$ and Assumption 5(1) that there exists some v_x such that $||v_x|| = 1$ and $v_x^T \nabla c_i(x^{k+1}) \leq -\delta_c$ for all $i \in \mathcal{A}(x^{k+1}; \theta_a)$, where $\mathcal{A}(x^{k+1}; \theta_a)$ is defined in (11). Let $\overline{\mathcal{A}}(x^{k+1}; \theta_a) = \{1, 2, \dots, \tilde{n}\} \setminus \mathcal{A}(x^{k+1}; \theta_a)$. Notice from (11) that $c_i(x^{k+1}) < -\theta_a$ for all $i \in \overline{\mathcal{A}}(x^{k+1}; \theta_a)$. In addition, observe from (72) that $\rho_k \geq \theta_a^{-1} \Lambda$. Using these and $||\lambda_x^k|| \leq \Lambda$, we obtain that $(\lambda_x^k + \rho_k c(x^{k+1}))_i \leq \Lambda - \rho_k \theta_a \leq 0$ for all $i \in \overline{\mathcal{A}}(x^{k+1}; \theta_a)$. By this and the fact that $v_x^T \nabla c_i(x^{k+1}) \leq -\delta_c$ for all $i \in \mathcal{A}(x^{k+1}; \theta_a)$, one has

$$v_{x}^{T} \nabla c(x^{k+1}) \tilde{\lambda}_{\mathbf{x}}^{k+1} \stackrel{(73)}{=} v_{x}^{T} \nabla c(x^{k+1}) [\lambda_{\mathbf{x}}^{k} + \rho_{k} c(x^{k+1})]_{+} = \sum_{i=1}^{\tilde{n}} v_{x}^{T} \nabla c_{i}(x^{k+1}) ([\lambda_{\mathbf{x}}^{k} + \rho_{k} c(x^{k+1})]_{+})_{i}$$

$$= \sum_{i \in \mathcal{A}(x^{k+1};\theta_{a})} v_{x}^{T} \nabla c_{i}(x^{k+1}) ([\lambda_{\mathbf{x}}^{k} + \rho_{k} c(x^{k+1})]_{+})_{i} + \sum_{i \in \bar{\mathcal{A}}(x^{k+1};\theta_{a})} v_{x}^{T} \nabla c_{i}(x^{k+1}) ([\lambda_{\mathbf{x}}^{k} + \rho_{k} c(x^{k+1})]_{+})_{i}$$

$$\leq -\delta_{c} \sum_{i \in \mathcal{A}(x^{k+1};\theta_{a})} ([\lambda_{\mathbf{x}}^{k} + \rho_{k} c(x^{k+1})]_{+})_{i} = -\delta_{c} \sum_{i=1}^{\tilde{n}} ([\lambda_{\mathbf{x}}^{k} + \rho_{k} c(x^{k+1})]_{+})_{i} \stackrel{(73)}{=} -\delta_{c} \|\tilde{\lambda}_{\mathbf{x}}^{k+1}\|_{1}.$$

$$(78)$$

Since (x^{k+1}, y^{k+1}) is an ϵ_k -stationary point of (9), it follows from (3) and (56) that there exists some $s \in \partial_x F(x^{k+1}, y^{k+1})$ such that

 $\overline{i=1}$

$$\|s + \nabla c(x^{k+1})[\lambda_{\mathbf{x}}^{k} + \rho_{k}c(x^{k+1})]_{+} - \nabla_{x}d(x^{k+1}, y^{k+1})[\lambda_{\mathbf{y}}^{k} + \rho_{k}d(x^{k+1}, y^{k+1})]_{+}\| \le \epsilon_{k},$$

which along with (73) and $\lambda_{\mathbf{y}}^{k+1} = [\lambda_{\mathbf{y}}^k + \rho_x d(x^{k+1}, y^{k+1})]_+$ implies that

$$\|s + \nabla c(x^{k+1})\tilde{\lambda}_{\mathbf{x}}^{k+1} - \nabla_x d(x^{k+1}, y^{k+1})\lambda_{\mathbf{y}}^{k+1}\| \le \epsilon_k.$$

By this, (78) and $||v_x|| = 1$, one has

$$\begin{aligned} \epsilon_{k} &\geq \|s + \nabla c(x^{k+1})\tilde{\lambda}_{\mathbf{x}}^{k+1} - \nabla_{x}d(x^{k+1}, y^{k+1})\lambda_{\mathbf{y}}^{k+1}\| \cdot \|v_{x}\| \\ &\geq \langle s + \nabla c(x^{k+1})\tilde{\lambda}_{\mathbf{x}}^{k+1} - \nabla_{x}d(x^{k+1}, y^{k+1})\lambda_{\mathbf{y}}^{k+1}, -v_{x} \rangle \\ &= -\langle s - \nabla_{x}d(x^{k+1}, y^{k+1})\lambda_{\mathbf{y}}^{k+1}, v_{x} \rangle - v_{x}^{T}\nabla c(x^{k+1})\tilde{\lambda}_{\mathbf{x}}^{k+1} \\ &\stackrel{(78)}{\geq} - \left(\|s\| + \|\nabla_{x}d(x^{k+1}, y^{k+1})\| \|\lambda_{\mathbf{y}}^{k+1}\| \right) \|v_{x}\| + \delta_{c} \|\tilde{\lambda}_{\mathbf{x}}^{k+1}\|_{1} \\ &\geq -L_{F} - L_{d} \|\lambda_{\mathbf{y}}^{k+1}\| + \delta_{c} \|\tilde{\lambda}_{\mathbf{x}}^{k+1}\|_{1}, \end{aligned}$$

where the last inequality is due to $||v_x|| = 1$ and Assumptions 1(i) and 1(iii). Notice from (72) that (44) holds. It then follows from (45) that $||\lambda_{\mathbf{y}}^{k+1}|| \leq 2\delta_d^{-1}(\epsilon_0 + L_F)D_{\mathbf{y}}$, which together with the above inequality and $\epsilon_k \leq \epsilon_0$ yields

$$\|\tilde{\lambda}_{\mathbf{x}}^{k+1}\| \le \|\tilde{\lambda}_{\mathbf{x}}^{k+1}\|_{1} \le \delta_{c}^{-1}(L_{F} + L_{d}\|\lambda_{\mathbf{y}}^{k+1}\| + \epsilon_{k}) \le \delta_{c}^{-1}(L_{F} + 2L_{d}\delta_{d}^{-1}(\epsilon_{0} + L_{F})D_{\mathbf{y}} + \epsilon_{0}).$$
(79)

By this and (73), one can observe that

$$\|[c(x^{k+1})]_{+}\| \le \rho_{k}^{-1}\|[\lambda_{\mathbf{x}}^{k} + \rho_{k}c(x^{k+1})]_{+}\| = \rho_{k}^{-1}\|\tilde{\lambda}_{\mathbf{x}}^{k+1}\| \le \rho_{k}^{-1}\delta_{c}^{-1}(L_{F} + 2L_{d}\delta_{d}^{-1}(\epsilon_{0} + L_{F})D_{\mathbf{y}} + \epsilon_{0}).$$

Hence, (74) holds as desired.

We next show that (75) holds. Indeed, by $\tilde{\lambda}_{\mathbf{x}}^{k+1} \geq 0$, (74) and (79), one has

$$\langle \tilde{\lambda}_{\mathbf{x}}^{k+1}, c(x^{k+1}) \rangle \leq \langle \tilde{\lambda}_{\mathbf{x}}^{k+1}, [c(x^{k+1})]_{+} \rangle \leq \| \tilde{\lambda}_{\mathbf{x}}^{k+1} \| \| [c(x^{k+1})]_{+} \|$$

$$\leq \rho_{k}^{-1} \delta_{c}^{-2} (L_{F} + 2L_{d} \delta_{d}^{-1} (\epsilon_{0} + L_{F}) D_{\mathbf{y}} + \epsilon_{0})^{2}.$$

$$(80)$$

Using a similar argument as for the proof of (47), we have

$$-\langle \tilde{\lambda}_{\mathbf{x}}^{k+1}, \rho_k^{-1} \lambda_{\mathbf{x}}^k \rangle \le \langle \tilde{\lambda}_{\mathbf{x}}^{k+1}, c(x^{k+1}) \rangle,$$

which along with $\|\lambda_{\mathbf{x}}^k\| \leq \Lambda$ and (79) yields

$$\langle \tilde{\lambda}_{\mathbf{x}}^{k+1}, c(x^{k+1}) \rangle \ge -\rho_k^{-1} \| \tilde{\lambda}_{\mathbf{x}}^{k+1} \| \| \lambda_{\mathbf{x}}^k \| \ge -\rho_k^{-1} \delta_c^{-1} (L_F + 2L_d \delta_d^{-1} (\epsilon_0 + L_F) D_{\mathbf{y}} + \epsilon_0) \Lambda.$$

The relation (75) then follows from this and (80).

We are now ready to prove Theorem 1.

Proof of Theorem 1. (i) Observe from the definition of K in (22) and $\epsilon_k = \epsilon_0 \tau^k$ that K is the smallest nonnegative integer such that $\epsilon_K \leq \varepsilon$. Hence, Algorithm 1 terminates and outputs (x^{K+1}, y^{K+1}) after K + 1 outer iterations. It follows from these and $\rho_k = \epsilon_k^{-1}$ that $\epsilon_K \leq \varepsilon$ and $\rho_K \geq \varepsilon^{-1}$. By this and (28), one can see that (50) and (72) holds for k = K. It then follows from Lemmas 4 and 7 that (29)-(34) hold.

(ii) Let K and N be given in (22) and (35). Recall from Lemma 6 that the number of evaluations of ∇f , ∇c , ∇d , proximal operator of p and q performed by Algorithm 3 at iteration k of Algorithm 1 is at most N_k , where N_k is given in (65). By this and statement (i) of this theorem, one can observe that the total number of evaluations of ∇f , ∇c , ∇d , proximal operator of p and q performed in Algorithm 1 is no more than $\sum_{k=0}^{K} N_k$, respectively. As a result, to prove statement (ii) of this theorem, it suffices to show that $\sum_{k=0}^{K} N_k \leq N$. Recall from (77)

and Algorithm 1 that $\rho_k L_c^2 \leq L_k \leq \rho_k \hat{L}$ and $\rho_k \geq 1 \geq \epsilon_k$. Using these, (24), (25), (26), (61), (62), (63) and (64), we obtain that

$$1 \ge \alpha_k \ge \min\left\{1, \sqrt{4\epsilon_k/(\rho_k D_\mathbf{y} \widehat{L})}\right\} \ge \epsilon_k^{1/2} \rho_k^{-1/2} \widehat{\alpha},\tag{81}$$

$$\delta_{k} \leq (2 + \epsilon_{k}^{-1/2} \rho_{k}^{1/2} \hat{\alpha}^{-1}) \rho_{k} \widehat{L} D_{\mathbf{x}}^{2} + \max\{1/D_{\mathbf{y}}, \rho_{k} \widehat{L}/4\} D_{\mathbf{y}}^{2} \leq \epsilon_{k}^{-1/2} \rho_{k}^{3/2} \hat{\delta},$$

$$(82)$$

$$16 \max\{1/(2\alpha_{k} L^{2}), 4/(\epsilon_{k}^{1/2} \rho_{k}^{-1/2} \hat{\alpha} \alpha_{k} L^{2})\} \rho_{k}$$

$$M_{k} \leq \frac{16 \max\left\{ \frac{1}{(2\rho_{k}L_{c}^{2}), \frac{4}{(\epsilon_{k}^{\prime} - \rho_{k}^{\prime} - \alpha\rho_{k}L_{c}^{2})}\right\} \rho_{k}}{\left[(3\rho_{k}\hat{L} + \frac{1}{(2D_{y})})^{2} / \min\{\rho_{k}L_{c}^{2}, \epsilon_{k}/(2D_{y})\} + 3\rho_{k}\hat{L} + \frac{1}{(2D_{y})} \right]^{-2} \epsilon_{k}^{2}} \times \left(\epsilon_{k}^{-1/2}\rho_{k}^{3/2}\hat{\delta} + 2\epsilon_{k}^{-1/2}\rho_{k}^{1/2}\hat{\alpha}^{-1} \left(F_{\mathrm{hi}} - F_{\mathrm{low}} + \frac{\Lambda^{2}}{2} + \frac{3}{2} \|\lambda_{y}^{0}\|^{2} + \frac{3(F_{\mathrm{hi}} - f_{\mathrm{low}}^{*} + D_{y}\epsilon_{0})}{1 - \tau} + \rho_{k}d_{\mathrm{hi}}^{2} + \frac{D_{y}}{4} + \rho_{k}\hat{L}D_{x}^{2} \right) \right)$$

$$(83)$$

$$\leq \frac{16\epsilon_{k}^{-1/2}\rho_{k}^{-1/2}\max\left\{1/(2L_{c}^{2}), 4/(\hat{\alpha}L_{c}^{2})\right\}\rho_{k}}{\epsilon_{k}^{2}\rho_{k}^{-4}\left[(3\hat{L}+1/(2D_{\mathbf{y}}))^{2}/\min\{L_{c}^{2}, 1/(2D_{\mathbf{y}})\}+3\hat{L}+1/(2D_{\mathbf{y}})\right]^{-2}\epsilon_{k}^{2}} \times (\epsilon_{k}^{-1/2}\rho_{k}^{3/2})\left(\hat{\delta}+2\hat{\alpha}^{-1}\right)\right] \\ \times \left(F_{\mathrm{hi}}-F_{\mathrm{low}}+\frac{\Lambda^{2}}{2}+\frac{3}{2}\|\lambda_{\mathbf{y}}^{0}\|^{2}+\frac{3(F_{\mathrm{hi}}-f_{\mathrm{low}}^{*}+D_{\mathbf{y}}\epsilon_{0})}{1-\tau}+d_{\mathrm{hi}}^{2}+\frac{D_{\mathbf{y}}}{4}+\hat{L}D_{\mathbf{x}}^{2}\right)\right) \leq \epsilon_{k}^{-5}\rho_{k}^{6}\widehat{M},$$

$$T_{k} \leq \left[16\left(L_{F}D_{\mathbf{y}}+F_{\mathrm{hi}}-f_{\mathrm{low}}^{*}+\Lambda+\frac{1}{2}(\tau^{-1}+\|\lambda_{\mathbf{y}}^{0}\|^{2})+\frac{F_{\mathrm{hi}}-f_{\mathrm{low}}^{*}+D_{\mathbf{y}}\epsilon_{0}}{1-\tau}+\frac{\Lambda^{2}}{2}+\frac{D_{\mathbf{y}}}{4}\right)\epsilon_{k}^{-2}\rho_{k}\widehat{L}\right] \\ +8(1+4D_{\mathbf{y}}^{2}\rho_{k}^{2}\widehat{L}^{2}\epsilon_{k}^{-2})\rho_{k}^{-1}-1\Big|_{+} \leq \epsilon_{k}^{-2}\rho_{k}\widehat{T},$$

where (83) follows from (24), (25), (26), (81), (82), $\rho_k L_c^2 \leq L_k \leq \rho_k \hat{L}$, and $\rho_k \geq 1 \geq \epsilon_k$. By the above inequalities, (65), (77), $\hat{T} \geq 1$ and $\rho_k \geq 1 \geq \epsilon_k$, one has

$$\begin{split} \sum_{k=0}^{K} N_{k} &\leq \sum_{k=0}^{K} \left(\left\lceil 96\sqrt{2} \left(1 + \left(24\rho_{k}\widehat{L} + 4/D_{\mathbf{y}} \right) / (\rho_{k}L_{c}^{2}) \right) \right\rceil + 2 \right) \max \left\{ 2, \sqrt{D_{\mathbf{y}}\rho_{k}\widehat{L}\epsilon_{k}^{-1}} \right\} \\ &\times \left((\epsilon_{k}^{-2}\rho_{k}\widehat{T} + 1) (\log(\epsilon_{k}^{-5}\rho_{k}^{6}\widehat{M}))_{+} + \epsilon_{k}^{-2}\rho_{k}\widehat{T} + 1 + 2\epsilon_{k}^{-2}\rho_{k}\widehat{T} \log(\epsilon_{k}^{-2}\rho_{k}\widehat{T} + 1) \right) \\ &\leq \sum_{k=0}^{K} \left(\left\lceil 96\sqrt{2} \left(1 + \left(24\widehat{L} + 4/D_{\mathbf{y}} \right) / L_{c}^{2} \right) \right\rceil + 2 \right) \max \left\{ 2, \sqrt{D_{\mathbf{y}}\widehat{L}} \right\} \epsilon_{k}^{-1/2}\rho_{k}^{1/2} \\ &\times \epsilon_{k}^{-2}\rho_{k} \left((\widehat{T} + 1) (\log(\epsilon_{k}^{-5}\rho_{k}^{6}\widehat{M}))_{+} + \widehat{T} + 1 + 2\widehat{T} \log(\epsilon_{k}^{-2}\rho_{k}\widehat{T} + 1) \right) \\ &\leq \sum_{k=0}^{K} \left(\left\lceil 96\sqrt{2} \left(1 + \left(24\widehat{L} + 4/D_{\mathbf{y}} \right) / L_{c}^{2} \right) \right\rceil + 2 \right) \max \left\{ 2, \sqrt{D_{\mathbf{y}}\widehat{L}} \right\} \\ &\times \epsilon_{k}^{-5/2}\rho_{k}^{3/2}\widehat{T} \left(2(\log(\epsilon_{k}^{-5}\rho_{k}^{6}\widehat{M}))_{+} + 2 + 2\log(2\epsilon_{k}^{-2}\rho_{k}\widehat{T}) \right) \\ &\leq \sum_{k=0}^{K} \left(\left\lceil 96\sqrt{2} \left(1 + \left(24\widehat{L} + 4/D_{\mathbf{y}} \right) / L_{c}^{2} \right) \right\rceil + 2 \right) \max \left\{ 2, \sqrt{D_{\mathbf{y}}\widehat{L}} \right\} \widehat{T} \\ &\times \epsilon_{k}^{-5/2}\rho_{k}^{3/2} \left(14\log\rho_{k} - 14\log\epsilon_{k} + 2(\log\widehat{M})_{+} + 2 + 2\log(2\widehat{T}) \right), \end{split}$$
(84)

By the definition of K in (22), one has $\tau^K \geq \tau \varepsilon / \epsilon_0$. Also, notice from Algorithm 1 that

 $\rho_k = \epsilon_k^{-1} = (\epsilon_0 \tau^k)^{-1}$. It then follows from these, (35) and (84) that

$$\begin{split} \sum_{k=0}^{K} N_k &\leq \sum_{k=0}^{K} \left(\left\lceil 96\sqrt{2} \left(1 + \left(24\widehat{L} + 4/D_y \right)/L_c^2 \right) \right\rceil + 2 \right) \max\left\{ 2, \sqrt{D_y \widehat{L}} \right\} \widehat{T} \\ &\times \epsilon_k^{-4} \left(28 \log(1/\epsilon_k) + 2(\log\widehat{M})_+ + 2 + 2\log(2\widehat{T}) \right) \\ &= \left(\left\lceil 96\sqrt{2} \left(1 + \left(24\widehat{L} + 4/D_y \right)/L_c^2 \right) \right\rceil + 2 \right) \max\left\{ 2, \sqrt{D_y \widehat{L}} \right\} \widehat{T} \\ &\times \sum_{k=0}^{K} \epsilon_0^{-4} \tau^{-4k} \left(28k \log(1/\tau) + 28 \log(1/\epsilon_0) + 2(\log\widehat{M})_+ + 2 + 2\log(2\widehat{T}) \right) \\ &\leq \left(\left\lceil 96\sqrt{2} \left(1 + \left(24\widehat{L} + 4/D_y \right)/L_c^2 \right) \right\rceil + 2 \right) \max\left\{ 2, \sqrt{D_y \widehat{L}} \right\} \widehat{T} \\ &\times \sum_{k=0}^{K} \epsilon_0^{-4} \tau^{-4k} \left(28K \log(1/\tau) + 28 \log(1/\epsilon_0) + 2(\log\widehat{M})_+ + 2 + 2\log(2\widehat{T}) \right) \\ &\leq \left(\left\lceil 96\sqrt{2} \left(1 + \left(24\widehat{L} + 4/D_y \right)/L_c^2 \right) \right\rceil + 2 \right) \max\left\{ 2, \sqrt{D_y \widehat{L}} \right\} \widehat{T} \epsilon_0^{-4} \\ &\times \tau^{-4K} (1 - \tau^4)^{-1} \left(28K \log(1/\tau) + 28 \log(1/\epsilon_0) + 2(\log\widehat{M})_+ + 2 + 2\log(2\widehat{T}) \right) \\ &\leq \left(\left\lceil 96\sqrt{2} \left(1 + \left(24\widehat{L} + 4/D_y \right)/L_c^2 \right) \right\rceil + 2 \right) \max\left\{ 2, \sqrt{D_y \widehat{L}} \right\} \widehat{T} \epsilon_0^{-4} (1 - \tau^4)^{-1} \\ &\times (\tau \varepsilon/\epsilon_0)^{-4} \left(28K \log(1/\tau) + 28 \log(1/\epsilon_0) + 2(\log\widehat{M})_+ + 2 + 2\log(2\widehat{T}) \right) \overset{(35)}{=} N, \end{split}$$

where the second last inequality is due to $\sum_{k=0}^{K} \tau^{-4k} \leq \tau^{-4K}/(1-\tau^4)$, and the last inequality is due to $\tau^K \geq \tau \varepsilon/\epsilon_0$. Hence, statement (ii) of this theorem holds as desired.

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A A first-order method for nonconvex-concave minimax problem

In this part we present a first-order method proposed in [26, Algorithm 2] for finding an ϵ -stationary point of the nonconvex-concave minimax problem

$$H^* = \min_{x} \max_{y} \left\{ H(x, y) \coloneqq h(x, y) + p(x) - q(y) \right\},$$
(85)

which has at least one optimal solution and satisfies the following assumptions.

- **Assumption 4.** (i) $p : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ and $q : \mathbb{R}^m \to \mathbb{R} \cup \{\infty\}$ are proper convex functions and continuous on dom p and dom q, respectively, and moreover, dom p and dom q are compact.
 - (ii) The proximal operator associated with p and q can be exactly evaluated.

(iii) h is $L_{\nabla h}$ -smooth on dom $p \times \text{dom } q$, and moreover, $h(x, \cdot)$ is concave for any $x \in \text{dom } p$.

For ease of presentation, we define

$$D_p = \max\{\|u - v\| | u, v \in \operatorname{dom} p\}, \quad D_q = \max\{\|u - v\| | u, v \in \operatorname{dom} q\},$$
(86)

$$H_{\text{low}} = \min\{H(x, y) | (x, y) \in \operatorname{dom} p \times \operatorname{dom} q\}.$$
(87)

Given an iterate (x^k, y^k) , the first-order method [26, Algorithm 2] finds the next iterate (x^{k+1}, y^{k+1}) by applying a modified optimal first-order method [26, Algorithm 1] to the strongly-convex-strongly-concave minimax problem

$$\min_{x} \max_{y} \left\{ h_k(x,y) = h(x,y) - \epsilon \|y - y^0\|^2 / (4D_q) + L_{\nabla h} \|x - x^k\|^2 \right\}.$$
(88)

For ease reference, we next present the modified optimal first-order method [26, Algorithm 1] in Algorithm 2 below for solving the strongly-convex-strongly-concave minimax problem

$$\min_{x} \max_{y} \left\{ \bar{h}(x,y) + p(x) - q(y) \right\},\tag{89}$$

where $\bar{h}(x, y)$ is σ_x -strongly-convex- σ_y -strongly-concave and $L_{\nabla \bar{h}}$ -smooth on dom $p \times \text{dom } q$ for some $\sigma_x, \sigma_y > 0$. In Algorithm 2, the functions \hat{h}, a_x^k and a_y^k are defined as follows:

$$\begin{split} \hat{h}(x,y) &= \bar{h}(x,y) - \sigma_x \|x\|^2 / 2 + \sigma_y \|y\|^2 / 2, \\ a_x^k(x,y) &= \nabla_x \hat{h}(x,y) + \sigma_x (x - \sigma_x^{-1} z_g^k) / 2, \quad a_y^k(x,y) = -\nabla_y \hat{h}(x,y) + \sigma_y y + \sigma_x (y - y_g^k) / 8, \end{split}$$

where y_g^k and z_g^k are generated at iteration k of Algorithm 2 below.

Algorithm 2 A modified optimal first-order method for problem (89)

Input: $\tau > 0, \ \bar{z}^0 = z_f^0 \in -\sigma_x \text{dom} p, \ \bar{y}^0 = y_f^0 \in \text{dom} q, \ (z^0, y^0) = (\bar{z}^0, \bar{y}^0), \ \bar{\alpha} =$ $\min\{1, \sqrt{8\sigma_y/\sigma_x}\}, \ \eta_z = \sigma_x/2, \ \eta_y = \min\{1/(2\sigma_y), 4/(\bar{\alpha}\sigma_x)\}, \ \beta_t = 2/(t+3), \ \zeta = 2/($ $\left(2\sqrt{5}(1+8L_{\nabla\bar{h}}/\sigma_x)\right)^{-1}, \gamma_x = \gamma_y = 8\sigma_x^{-1}, \text{ and } \hat{\zeta} = \min\{\sigma_x, \sigma_y\}/L_{\nabla\bar{h}}^2.$ 1: for $k = 0, 1, 2, \dots$ do $(z_g^k, y_g^k) = \bar{\alpha}(z^k, y^k) + (1 - \bar{\alpha})(z_f^k, y_f^k).$ 2: $\begin{aligned} (x^{k,-1}, y^{k,-1}) &= (-\sigma_x^{-1} z_g^k, y_g^k), \\ x^{k,0} &= \operatorname{prox}_{\zeta \gamma_x p} (x^{k,-1} - \zeta \gamma_x a_x^k (x^{k,-1}, y^{k,-1})). \end{aligned}$ 3: 4: $y^{k,0} = \operatorname{prox}_{\zeta\gamma_y q}(y^{k,-1} - \zeta\gamma_y a_y^k(x^{k,-1}, y^{k,-1})).$ 5: $b_x^{k,0} = \frac{1}{\zeta \gamma_x} (x^{k,-1} - \zeta \gamma_x a_x^k (x^{k,-1}, y^{k,-1}) - x^{k,0})$ 6: $b_y^{k,0} = \frac{1}{\zeta \gamma_y} (y^{k,-1} - \zeta \gamma_y a_y^k (x^{k,-1}, y^{k,-1}) - y^{k,0}).$ 7:8: t = 0.while 9: $\gamma_{x} \|a_{x}^{k}(x^{k,t}, y^{k,t}) + b_{x}^{k,t}\|^{2} + \gamma_{y} \|a_{y}^{k}(x^{k,t}, y^{k,t}) + b_{y}^{k,t}\|^{2} > \gamma_{x}^{-1} \|x^{k,t} - x^{k,-1}\|^{2} + \gamma_{y}^{-1} \|y^{k,t} - y^{k,-1}\|^{2}$ $x^{k,t+1/2} = x^{k,t} + \beta_t (x^{k,0} - x^{k,t}) - \zeta \gamma_x (a_x^k (x^{k,t}, y^{k,t}) + b_x^{k,t}).$ 10: $y^{k,t+1/2} = y^{k,t} + \beta_t(y^{k,0} - y^{k,t}) - \zeta \gamma_u(a_u^k(x^{k,t}, y^{k,t}) + b_u^{k,t}).$ 11: $x^{k,t+1} = \operatorname{prox}_{\zeta\gamma_x p}(x^{k,t} + \beta_t(x^{k,0} - x^{k,t}) - \zeta\gamma_x a_x^k(x^{k,t+1/2}, y^{k,t+1/2})).$ 12: $y^{k,t+1} = \operatorname{prox}_{\zeta\gamma_y q}(y^{k,t} + \beta_t(y^{k,0} - y^{k,t}) - \zeta\gamma_y a_y^k(x^{k,t+1/2}, y^{k,t+1/2})).$ 13: $b_x^{k,t+1} = \frac{1}{\zeta \gamma_x} (x^{k,t} + \beta_t (x^{k,0} - x^{k,t}) - \zeta \gamma_x a_x^k (x^{k,t+1/2}, y^{k,t+1/2}) - x^{k,t+1})$ 14: $b_{y}^{k,t+1} = \frac{1}{\zeta \gamma_{y}} (y^{k,t} + \beta_{t} (y^{k,0} - y^{k,t}) - \zeta \gamma_{y} a_{y}^{k} (x^{k,t+1/2}, y^{k,t+1/2}) - y^{k,t+1}).$ 15: $t \leftarrow t + 1$ 16:end while 17: $(x_f^{k+1}, y_f^{k+1}) = (x^{k,t}, y^{k,t}).$ 18: $\begin{aligned} &(z_f^{k+1}, w_f^{k+1}) = (\nabla_x \hat{h}(x_f^{k+1}, y_f^{k+1}) + b_x^{k,t}, -\nabla_y \hat{h}(x_f^{k+1}, y_f^{k+1}) + b_y^{k,t}), \\ &z^{k+1} = z^k + \eta_z \sigma_x^{-1}(z_f^{k+1} - z^k) - \eta_z(x_f^{k+1} + \sigma_x^{-1} z_f^{k+1}), \\ &y^{k+1} = y^k + \eta_y \sigma_y(y_f^{k+1} - y^k) - \eta_y(w_f^{k+1} + \sigma_y y_f^{k+1}). \end{aligned}$ 19:20: 21: $x^{k+1} = -\sigma_r^{-1} z^{k+1}$ 22: $\tilde{x}^{k+1} = \operatorname{prox}_{\hat{\zeta}_p}(x^{k+1} - \hat{\zeta} \nabla_x \bar{h}(x^{k+1}, y^{k+1})).$ 23: $\tilde{y}^{k+1} = \operatorname{prox}_{\hat{\zeta}_a}(y^{k+1} + \hat{\zeta} \nabla_y \bar{h}(x^{k+1}, y^{k+1})).$ 24: Terminate the algorithm and output $(\tilde{x}^{k+1}, \tilde{y}^{k+1})$ if 25: $\|\hat{\zeta}^{-1}(x^{k+1} - \tilde{x}^{k+1}, \tilde{y}^{k+1} - u^{k+1}) - (\nabla \bar{h}(x^{k+1}, u^{k+1}) - \nabla \bar{h}(\tilde{x}^{k+1}, \tilde{y}^{k+1}))\| < \tau.$

26: **end for**

We are now ready to present the first-order method [26, Algorithm 2] for finding an ϵ -stationary point of (85) in Algorithm 3 below.

⁴For convenience, $-\sigma_x \operatorname{dom} p$ stands for the set $\{-\sigma_x u | u \in \operatorname{dom} p\}$.

Algorithm 3 A first-order method for problem (85)

Input: $\epsilon > 0, \epsilon_0 \in (0, \epsilon/2], (\hat{x}^0, \hat{y}^0) \in \text{dom } p \times \text{dom } q, (x^0, y^0) = (\hat{x}^0, \hat{y}^0), \text{ and } \epsilon_k = \epsilon_0/(k+1).$ 1: for k = 0, 1, 2, ... do

- 2: Call Algorithm 2 with $\bar{h} \leftarrow h_k, \tau \leftarrow \epsilon_k, \sigma_x \leftarrow L_{\nabla h}, \sigma_y \leftarrow \epsilon/(2D_q), L_{\nabla \bar{h}} \leftarrow 3L_{\nabla h} + \epsilon/(2D_q),$ $\bar{z}^0 = z_f^0 \leftarrow -\sigma_x x^k, \ \bar{y}^0 = y_f^0 \leftarrow y^k,$ and denote its output by $(x^{k+1}, y^{k+1}),$ where h_k is given in (88).
- 3: Terminate the algorithm and output $(x_{\epsilon}, y_{\epsilon}) = (x^{k+1}, y^{k+1})$ if

$$\|x^{k+1} - x^k\| \le \epsilon/(4L_{\nabla h}).$$

4: end for

The following theorem presents the iteration complexity of Algorithm 3, whose proof is given in [26, Theorem 2].

Theorem 2 (Complexity of Algorithm 3). Suppose that Assumption 4 holds. Let H^* , $H D_p$, D_q , and H_{low} be defined in (85), (86) and (87), $L_{\nabla h}$ be given in Assumption 4, ϵ , ϵ_0 and x^0 be given in Algorithm 3, and

$$\begin{aligned} \alpha &= \min\left\{1, \sqrt{4\epsilon/(D_q L_{\nabla h})}\right\}, \\ \delta &= (2 + \alpha^{-1})L_{\nabla h}D_p^2 + \max\left\{\epsilon/D_q, \alpha L_{\nabla h}/4\right\}D_q^2, \\ K &= \left[16(\max_y H(x^0, y) - H^* + \epsilon D_q/4)L_{\nabla h}\epsilon^{-2} + 32\epsilon_0^2(1 + 4D_q^2 L_{\nabla h}^2 \epsilon^{-2})\epsilon^{-2} - 1\right]_+, \\ N &= \left(\left[96\sqrt{2}\left(1 + (24L_{\nabla h} + 4\epsilon/D_q)L_{\nabla h}^{-1}\right)\right] + 2\right)\left\{2, \sqrt{D_q L_{\nabla h}}\epsilon^{-1}\right\} \\ &\times \left((K + 1)\left(\log\frac{4\max\left\{\frac{1}{2L_{\nabla h}}, \min\left\{\frac{D_q}{\epsilon}, \frac{4}{\alpha L_{\nabla h}}\right\}\right\}\left(\delta + 2\alpha^{-1}(H^* - H_{\text{low}} + \epsilon D_q/4 + L_{\nabla h}D_p^2)\right)}{[(3L_{\nabla h} + \epsilon/(2D_q))^2/\min\{L_{\nabla h}, \epsilon/(2D_q)\} + 3L_{\nabla h} + \epsilon/(2D_q)]^{-2}\epsilon_0^2}\right)_+ \\ &+ K + 1 + 2K\log(K + 1)\right). \end{aligned}$$

Then Algorithm 3 terminates and outputs an ϵ -stationary point $(x_{\epsilon}, y_{\epsilon})$ of (85) in at most K+1 outer iterations that satisfies

$$\max_{y} H(x_{\epsilon}, y) \leq \max_{y} H(\hat{x}^{0}, y) + \epsilon D_{q}/4 + 2\epsilon_{0}^{2} \left(L_{\nabla h}^{-1} + 4D_{q}^{2}L_{\nabla h}\epsilon^{-2} \right).$$

Moreover, the total number of evaluations of ∇h and proximal operator of p and q performed in Algorithm 3 is no more than N, respectively.