Iterative Hard Thresholding Methods for l_0 Regularized Convex Cone Programming

Zhaosong Lu[*]

October 30, 2012 (Revised: August 15, 2013)

Abstract

In this paper we consider l_0 regularized convex cone programming problems. In particular, we first propose an iterative hard thresholding (IHT) method and its variant for solving l_0 regularized box constrained convex programming. We show that the sequence generated by these methods converges to a local minimizer. Also, we establish the iteration complexity of the IHT method for finding an ϵ-local-optimal solution. We then propose a method for solving l_0 regularized convex cone programming by applying the IHT method to its quadratic penalty relaxation and establish its iteration complexity for finding an ϵ-approximate local minimizer. Finally, we propose a variant of this method in which the associated penalty parameter is dynamically updated, and show that every accumulation point is a local minimizer of the problem.

Key words: Sparse approximation, iterative hard thresholding method, l_0 regularization, box constrained convex programming, convex cone programming

1 Introduction

Sparse approximations have over the last decade gained a great deal of popularity in numerous areas. For example, in compressed sensing [8, 11], a large sparse signal is decoded by finding a sparse solution to a system of linear equalities and/or inequalities. The similar ideas have also been widely used in linear regression. Recently, sparse inverse covariance selection becomes an important tool in discovering the conditional independence in graphical models. One popular approach for sparse inverse covariance selection is to find an approximate sparse inverse covariance while maximizing the log-likelihood (see, for example, [10]). Similarly, sparse logistic regression has been proposed as a promising method for feature selection in

classification problems in which a sparse solution is sought to minimize the average logistic loss (see, for example, [20]). The common point of these applications is to find a sparse approximation to a specific instance of a generic class of convex cone programming problems:

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & Ax - b \in \mathcal{K}^*, \\ & l \le x \le u \end{aligned} \tag{1}$$

for some $l \in \bar{\Re}_-^n$, $u \in \bar{\Re}_+^n$, $A \in \Re^{m \times n}$ and $b \in \Re^m$, where $\mathcal{K}^*$ denotes the dual cone of a closed convex cone $\mathcal{K} \subseteq \Re^m$, i.e., $\mathcal{K}^* = \{s \in \Re^m : s^T x \ge 0, \forall x \in \mathcal{K}\}$, and $\bar{\Re}_-^n = \{x : -\infty \le x_i \le 0, \ 1 \le i \le n\}$ and $\bar{\Re}_+^n = \{x \in \Re^n : 0 \le x_i \le \infty, \ 1 \le i \le n\}$. A sparse solution to (1) can be sought by solving the following l_0 regularized convex cone programming problem:

$$\begin{aligned} \min \quad & f(x) + \lambda \|x\|_0 \\ \text{s.t.} \quad & Ax - b \in \mathcal{K}^*, \\ & l \le x \le u \end{aligned} \tag{2}$$

for some $\lambda > 0$, where $\|x\|_0$ denotes the cardinality of x. One special case of (2) with $\mathcal{K} = \{0\}$, $f(x) = \|Cx - d\|^2$, $l_i = -\infty$ and $u_i = \infty$ for all i, that is, the l_0-regularized unconstrained least squares problem, has been well studied in the literature (e.g., [21, 16]), and some methods were developed for solving it. For example, the iterative hard thresholding (IHT) methods [13, 5, 6] were proposed to solve this type of problems and the related l_0-constrained problems, respectively. Moreover, Blumensath and Davies [5] showed that the IHT method converges to a local minimizer of l_0-regularized and -constrained least squares problems. In addition, matching pursuit algorithms [18, 23] and some variants of IHT method [17, 7, 9, 12] were studied for solving the l_0-constrained least squares problems in the context of compressed sensing. Recently, several approaches have been proposed to solve some sparsity-constrained recovery problems with nonlinear observations (see, for example, [4, 1]). Also, Lu and Zhang [16] proposed a penalty decomposition method for solving a more general class of l_0-regularized and -constrained minimization problems.

As shown by the extensive experiments in [5, 6], the IHT method performs very well in finding a sparse solution to unconstrained least squares problems. In addition, the similar type of methods [14, 22] were successfully applied to find low rank solutions in the context of matrix completion. Inspired by these works, in this paper we study IHT methods for solving l_0 regularized convex cone programming problem (2). In particular, we first propose an IHT method and its variant for solving l_0 regularized box constrained convex programming. We show that the sequence generated by these methods converges to a local minimizer. [1] Also, we establish the iteration complexity of the IHT method for finding an ϵ-local-optimal solution. We then propose a method for solving l_0 regularized convex cone programming by applying the IHT method to its quadratic penalty relaxation and establish its iteration complexity for finding an ϵ-approximate local minimizer of the problem. We also propose a variant of the

---

[1]Such a convergence result of the IHT method when applied to l_0-regularized and -constrained least squares problems was already established by Blumensath and Davies [5].

method in which the associated penalty parameter is dynamically updated, and show that every accumulation point is a local minimizer of the problem.

The outline of this paper is as follows. In Subsection 1.1 we introduce some notations that are used in the paper. In Section 2 we present some technical results about a projected gradient method for convex programming. In Section 3 we propose IHT methods for solving l_0 regularized box constrained convex programming and study their convergence. In section 4 we develop IHT methods for solving l_0 regularized convex cone programming and study their convergence. Finally, in Section 5 we present some concluding remarks.

1.1 Notation

Given a nonempty closed convex $\Omega \subseteq \Re^n$ and an arbitrary point $x \in \Omega$, $\mathcal{N}_\Omega(x)$ denotes the normal cone of Ω at x. In addition, $d_\Omega(y)$ denotes the Euclidean distance between $y \in \Re^n$ and Ω. All norms used in the paper are Euclidean norm denoted by $\|\cdot\|$. We use $\mathcal{U}(r)$ to denote a ball centered at the origin with a radius $r \geq 0$, that is, $\mathcal{U}(r) := \{x \in \Re^n : \|x\| \leq r\}$.

2 Technical preliminaries

In this section we present some technical results about a projected gradient method for convex programming that will be subsequently used in this paper.

Consider the convex programming problem

$$\phi^* := \min_{x \in X} \phi(x), \tag{3}$$

where $X \subseteq \Re^n$ is a closed convex set and $\phi : X \to \Re$ is a smooth convex function whose gradient is Lipschitz continuous with constant $L_\phi > 0$. Assume that the set of optimal solutions of (3), denoted by X^*, is nonempty.

Let $L \geq L_\phi$ be arbitrarily given. A projected gradient of ϕ at any $x \in X$ with respect to X is defined as

$$g(x) := L\left[x - \Pi_X\left(x - \nabla\phi(x)/L\right)\right], \tag{4}$$

where $\Pi_X(\cdot)$ is the projection map onto X (see, for example, [19]).

The following properties of the projected gradient can be directly obtained from Proposition 3 and Lemma 4 of [15] by replacing τ and $\nabla\phi(\cdot)]_X^\tau$ by $1/L$ and $g(\cdot)$, respectively.

Lemma 2.1 *Let $x \in X$ be given and define $x^+ := \Pi_X(x - \nabla\phi(x)/L)$. Then, for any given $\epsilon \geq 0$, the following statements hold:*

a) $\|g(x)\| \leq \epsilon$ *if and only if* $\nabla\phi(x) \in -\mathcal{N}_X(x^+) + \mathcal{U}(\epsilon)$.

b) $\|g(x)\| \leq \epsilon$ *implies that* $\nabla\phi(x^+) \in -\mathcal{N}_X(x^+) + \mathcal{U}(2\epsilon)$.

c) $\phi(x^+) - \phi(x) \leq -\|g(x)\|^2/(2L)$.

d) $\phi(x) - \phi(x^*) \geq \|g(x)\|^2/(2L)$, where $x^* \in \operatorname{Arg\,min}\{\phi(y) : y \in X\}$.

We next study a projected gradient method for solving (3).

Projected gradient method for (3):

Choose an arbitrary $x^0 \in X$. Set $k = 0$.

1) Solve the subproblem

$$x^{k+1} = \arg\min_{x \in X}\{\phi(x^k) + \nabla\phi(x^k)^T(x - x^k) + \frac{L}{2}\|x - x^k\|^2\}. \qquad (5)$$

2) Set $k \leftarrow k + 1$ and go to step 1).

end

The convergence properties of the above projected gradient method are established in the following two theorems, which will be used in the subsequent sections of this paper. The first theorem shows that the method is sublinearly convergent for a convex function with Lipschitz gradient. And the second theorem shows that the method is linearly convergent for a smooth strongly convex function.

Theorem 2.2 *Let $\{x^k\}$ be generated by the above projected gradient method. Then the following statements hold:*

(i) *For every $k \geq 0$ and $l \geq 1$,*

$$\phi(x^{k+l}) - \phi^* \leq \frac{L}{2l}\|x^k - x^*\|^2. \qquad (6)$$

(ii) *$\{x^k\}$ converges to some optimal solution x^* of (3).*

Proof. (i) Since the objective function of (5) is strongly convex with modulus L, it follows that for every $x \in X$,

$$\phi(x^k)+\nabla\phi(x^k)^T(x-x^k)+\frac{L}{2}\|x-x^k\|^2 \geq \phi(x^k)+\nabla\phi(x^k)^T(x^{k+1}-x^k)+\frac{L}{2}\|x^{k+1}-x^k\|^2+\frac{L}{2}\|x-x^{k+1}\|^2.$$

By the convexity of ϕ, Lipschitz continuity of $\nabla\phi$ and $L \geq L_\phi$, we have

$$\begin{aligned}
\phi(x) &\geq \phi(x^k) + \nabla\phi(x^k)^T(x - x^k), \\
\phi(x^{k+1}) &\leq \phi(x^k) + \nabla\phi(x^k)^T(x^{k+1} - x^k) + \frac{L}{2}\|x^{k+1} - x^k\|^2,
\end{aligned}$$

which together with the above inequality imply that

$$\phi(x) + \frac{L}{2}\|x - x^k\|^2 \geq \phi(x^{k+1}) + \frac{L}{2}\|x - x^{k+1}\|^2, \quad \forall x \in X. \qquad (7)$$

4

Letting $x = x^k$ in (7), we obtain that

$$\phi(x^k) - \phi(x^{k+1}) \ \geq \ L\|x^{k+1} - x^k\|^2/2.$$

Hence, $\{\phi(x^k)\}$ is decreasing. Letting $x = x^* \in X^*$ in (7), we have

$$\phi(x^{k+1}) - \phi^* \ \leq \ \frac{L}{2}\left(\|x^k - x^*\|^2 - \|x^{k+1} - x^*\|^2\right), \quad \forall k \geq 0.$$

Using this inequality and the monotonicity of $\{\phi(x^k)\}$, we obtain that

$$l(\phi(x^{k+l}) - \phi^*) \ \leq \ \sum_{i=k}^{k+l-1}[\phi(x^{i+1}) - \phi^*] \ \leq \ \frac{L}{2}\left(\|x^k - x^*\|^2 - \|x^{k+l} - x^*\|^2\right), \tag{8}$$

which immediately yields (6).

(ii) It follows from (8) that

$$\|x^{k+l} - x^*\| \ \leq \ \|x^k - x^*\|, \quad \forall k \geq 0, l \geq 1. \tag{9}$$

Hence, $\|x^k - x^*\| \leq \|x^0 - x^*\|$ for every k. It implies that $\{x^k\}$ is bounded. Then, there exists a subsequence K such that $\{x^k\}_K \to \hat{x}^* \in X$. It can be seen from (6) that $\{\phi(x^k)\}_K \to \phi^*$. Hence, $\phi(\hat{x}^*) = \lim_{k \in K \to \infty} \phi(x^k) = \phi^*$, which implies that $\hat{x}^* \in X^*$. Since (9) holds for any $x^* \in X^*$, we also have $\|x_{k+l} - \hat{x}^*\| \ \leq \ \|x^k - \hat{x}^*\|$ for every $k \geq 0$ and $l \geq 1$. This together with the fact $\{x^k\}_K \to \hat{x}^*$ implies that $\{x^k\} \to \hat{x}^*$ and hence statement (ii) holds. ∎

Theorem 2.3 *Suppose that ϕ is strongly convex with modulus $\sigma > 0$. [2] Let $\{x^k\}$ be generated by the above projected gradient method. Then, for any given $\epsilon > 0$, the following statements hold:*

(i) $\phi(x^k) - \phi^* \leq \epsilon$ *whenever*

$$k \geq 2\lceil L/\sigma\rceil\left\lceil \log\frac{\phi(x^0) - \phi^*}{\epsilon}\right\rceil.$$

(ii) $\phi(x^k) - \phi^* < \epsilon$ *whenever*

$$k \geq 2\lceil L/\sigma\rceil\left\lceil \log\frac{\phi(x^0) - \phi^*}{\epsilon}\right\rceil + 1.$$

Proof. (i) Let $M = \lceil L/\sigma\rceil$. It follows from Theorem 2.2 that

$$\phi(x^{k+l}) - \phi^* \leq \frac{L}{2l}\|x^k - x^*\|^2 \leq \frac{L}{\sigma l}(\phi(x^k) - \phi^*),$$

---

[2]ϕ is strongly convex with modulus $\sigma > 0$ if $\phi(\cdot) - \frac{\sigma}{2}\|\cdot\|^2$ is convex.

where x^* is the optimal solution of (3). Hence, we have

$$\phi(x^{k+2M}) - \phi^* \le \frac{L}{2\sigma M}(\phi(x^k) - \phi^*) \le \frac{1}{2}(\phi(x^k) - \phi^*),$$

which implies that

$$\phi(x^{2jM}) - \phi^* \le \frac{1}{2^j}(\phi(x^0) - \phi^*).$$

Let $K = \lceil \log((\phi(x^0) - \phi^*)/\epsilon) \rceil$. Hence, when $k \ge 2KM$, we have

$$\phi(x^k) - \phi^* \le \phi(x^{2KM}) - \phi^* \le \frac{1}{2^K}(\phi(x^0) - \phi^*) \le \epsilon,$$

which immediately implies that statement (i) holds.

(ii) Let K and M be defined as above. If $\phi(x^{2KM}) = \phi^*$, by monotonicity of $\{\phi(x^k)\}$ we have $\phi(x^k) = \phi^*$ when $k > 2KM$, and hence the conclusion holds. We now suppose that $\phi(x^{2KM}) > \phi^*$. It implies that $g(x^{2KM}) \ne 0$, where g is defined in (4). Using this relation, Lemma 2.1 (c) and statement (i), we obtain that $\phi(x^{2KM+1}) < \phi(x^{2KM}) \le \epsilon$, which together with the monotonicity of $\{\phi(x^k)\}$ implies that the conclusion holds. ∎

Finally, we consider the convex programming problem

$$f^* := \min\{f(x) : Ax - b \in \mathcal{K}^*, x \in X\}, \tag{10}$$

for some $A \in \Re^{m \times n}$ and $b \in \Re^m$, where $f : X \to \Re$ is a smooth convex function whose gradient is Lipschitz continuous gradient with constant $L_f > 0$, $X \subseteq \Re^n$ is a closed convex set, and $\mathcal{K}^*$ is the dual cone of a closed convex cone $\mathcal{K}$.

The Lagrangian dual function associated with (10) is given by

$$d(\mu) := \inf\{f(x) + \mu^T(Ax - b) : x \in X\}, \quad \forall \mu \in -\mathcal{K}.$$

Assume that there exists a Lagrange multiplier for (10), that is, a vector $\mu^* \in -\mathcal{K}$ such that $d(\mu^*) = f^*$. Under this assumption, the following results are established in Corollary 2 and Proposition 10 of [15], respectively.

Lemma 2.4 *Let μ^* be a Lagrange multiplier for* (10). *There holds:*

$$f(x) - f^* \ge -\|\mu^*\| d_{\mathcal{K}^*}(Ax - b), \quad \forall x \in X.$$

Lemma 2.5 *Let $\rho > 0$ be given and $L_\rho = L_f + \rho\|A\|^2$. Consider the problem*

$$\Phi_\rho^* := \min_{x \in X}\{\Phi_\rho(x) := f(x) + \frac{\rho}{2}[d_{\mathcal{K}^*}(Ax - b)]^2\}. \tag{11}$$

If $x \in X$ is a ξ-approximate solution of (11), *i.e., $\Phi_\rho(x) - \Phi_\rho^* \le \xi$, then the pair (x^+, μ) defined as*

$$\begin{aligned}
x^+ &:= \Pi_X(x - \nabla\Phi_\rho(x)/L_\rho), \\
\mu &:= \rho[Ax^+ - b - \Pi_{\mathcal{K}^*}(Ax^+ - b)]
\end{aligned}$$

is in $X \times (-\mathcal{K})$ and satisfies $\mu^T \Pi_{\mathcal{K}^*}(Ax^+ - b) = 0$ and the relations

$$
\begin{aligned}
d_{\mathcal{K}^*}(Ax^+ - b) &\leq \frac{1}{\rho}\|\mu^*\| + \sqrt{\frac{\xi}{\rho}}, \\
\nabla f(x^+) + A^T \mu &\in -\mathcal{N}_X(x^+) + \mathcal{U}(2\sqrt{2L_\rho\xi}),
\end{aligned}
$$

where μ^* is an arbitrary Lagrange multiplier for (10).

3 l_0 regularized box constrained convex programming

In this section we consider a special case of (2) with $\mathcal{K} = \{0\}$, that is, l_0 regularized box constrained convex programming problem in the form of:

$$
\begin{aligned}
\underline{F}^* := \min \quad & F(x) := f(x) + \lambda\|x\|_0 \\
\text{s.t.} \quad & l \leq x \leq u
\end{aligned}
\tag{12}
$$

for some $\lambda > 0$, $l \in \bar{\Re}_-^n$ and $u \in \bar{\Re}_+^n$. Recently, Blumensath and Davies [5, 6] proposed an iterative hard thresholding (IHT) method for solving a special case of (12) with $f(x) = \|Cx - d\|^2$, $l_i = -\infty$ and $u_i = \infty$ for all i. Our aim is to extend their IHT method to solve (12) and study its convergence. In addition, we establish its iteration complexity for finding an ϵ-local-optimal solution of (12). Finally, we propose a variant of the IHT method in which only "local" Lipschitz constant of ∇f is used.

Throughout this section we assume that f is a smooth convex function in $\mathcal{B}$ whose gradient is Lipschitz continuous with constant $L_f > 0$, and also that f is bounded below on the set $\mathcal{B}$, where

$$
\mathcal{B} := \{x \in \Re^n : l \leq x \leq u\}.
\tag{13}
$$

We now present an IHT method for solving problem (12).

Iterative hard thresholding method for (12):

Choose an arbitrary $x^0 \in \mathcal{B}$. Set $k = 0$.

   1) Solve the subproblem

$$
x^{k+1} \in \operatorname*{Arg\,min}_{x \in \mathcal{B}}\{f(x^k) + \nabla f(x^k)^T(x - x^k) + \frac{L}{2}\|x - x^k\|^2 + \lambda\|x\|_0\}.
\tag{14}
$$

   2) Set $k \leftarrow k + 1$ and go to step 1).

end

   Remark. The subproblem (14) has a closed form solution (see Lemma 3.2).   ■

   In what follows, we study the convergence of the above IHT method for (12). Before

proceeding, we introduce some notations that will be used subsequently. Define

$$\mathcal{B}_I \quad := \quad \{x \in \mathcal{B} : x_I = 0\}, \quad \forall I \subseteq \{1, \ldots, n\}, \tag{15}$$

$$\Pi_{\mathcal{B}}(x) \quad := \quad \arg\min\{\|y - x\| : y \in \mathcal{B}\}, \quad \forall x \in \Re^n, \tag{16}$$

$$s_L(x) \quad := \quad x - \frac{1}{L}\nabla f(x), \quad \forall x \in \mathcal{B}, \tag{17}$$

$$I(x) \quad := \quad \{i : x_i = 0\}, \quad \forall x \in \Re^n \tag{18}$$

for some constant $L > L_f$.

The following lemma establishes some properties of the operators $s_L(\cdot)$ and $\Pi_{\mathcal{B}}(s_L(\cdot))$, which will be used subsequently.

Lemma 3.1 *For any x, $y \in \Re^n$, there hold:*

(1) $|[s_L(x)]_i^2 - [s_L(y)]_i^2| \le 4(\|x - y\| + |[s_L(y)]_i|)\|x - y\|;$

(2) $|[\Pi_{\mathcal{B}}(s_L(x)) - s_L(x)]_i^2 - [\Pi_{\mathcal{B}}(s_L(y)) - s_L(y)]_i^2| \le 4(\|x-y\| + |[\Pi_{\mathcal{B}}(s_L(y)) - s_L(y)]_i|)\|x-y\|.$

Proof. (1) We observe that

$$\|s_L(x) - s_L(y)\| \quad = \quad \|x - y - \frac{1}{L}(\nabla f(x) - \nabla f(y))\| \quad \le \quad \|x - y\| + \frac{1}{L}\|\nabla f(x) - \nabla f(y)\|,$$

$$\le \quad (1 + \frac{L_f}{L})\|x - y\| \quad \le \quad 2\|x - y\|. \tag{19}$$

It follows from (19) that

$$|[s_L(x)]_i^2 - [s_L(y)]_i^2| \quad = \quad |[s_L(x)]_i + [s_L(y)]_i| \cdot |[s_L(x)]_i - [s_L(y)]_i|,$$

$$\le \quad (|[s_L(x)]_i - [s_L(y)]_i| + 2|[s_L(y)]_i|) \cdot |[s_L(x)]_i - [s_L(y)]_i|,$$

$$\le \quad 4(\|x - y\| + |[s_L(y)]_i|)\|x - y\|.$$

(2) It can be shown that

$$\|\Pi_{\mathcal{B}}(x) - x + y - \Pi_{\mathcal{B}}(y)\| \quad \le \quad \|x - y\|.$$

Using this inequality and (19), we then have

$$|[\Pi_{\mathcal{B}}(s_L(x)) - s_L(x)]_i^2 - [\Pi_{\mathcal{B}}(s_L(y)) - s_L(y)]_i^2|$$
$$\le (|[\Pi_{\mathcal{B}}(s_L(x)) - s_L(x)]_i - [\Pi_{\mathcal{B}}(s_L(y)) - s_L(y)]_i| + 2|\Pi_{\mathcal{B}}(s_L(y)) - s_L(y)]_i|)$$
$$\quad \cdot |[\Pi_{\mathcal{B}}(s_L(x)) - s_L(x)]_i - [\Pi_{\mathcal{B}}(s_L(y)) - s_L(y)]_i|,$$
$$\le (\|s_L(x) - s_L(y)\| + 2|[\Pi_{\mathcal{B}}(s_L(y)) - s_L(y)]_i|) \cdot \|s_L(x) - s_L(y)\|,$$
$$\le 4(\|x - y\| + |[\Pi_{\mathcal{B}}(s_L(y)) - s_L(y)]_i|)\|x - y\|.$$

$\blacksquare$

The following lemma shows that subproblem (14) has a closed-form solution.

8

Lemma 3.2 *The solution x^{k+1} of subproblem* (14) *is given as follows: for $i = 1, \ldots, n$,*

$$
x_i^{k+1} = \begin{cases} [\Pi_{\mathcal{B}}(s_L(x^k))]_i, & \text{if } [s_L(x^k)]_i^2 - [\Pi_{\mathcal{B}}(s_L(x^k)) - s_L(x^k)]_i^2 > \frac{2\lambda}{L}, \\ 0, & \text{if } [s_L(x^k)]_i^2 - [\Pi_{\mathcal{B}}(s_L(x^k)) - s_L(x^k)]_i^2 < \frac{2\lambda}{L}, \\ [\Pi_{\mathcal{B}}(s_L(x^k))]_i \text{ or } 0, & \text{otherwise.} \end{cases} \tag{20}
$$

Proof. Let $i \in \{1, \ldots, n\}$ be arbitrarily chosen. One can observe that the objective function and the constrained set of (14) are both separable. Using this fact, (17), and dropping some constant, we see that

$$
x_i^{k+1} \in \mathrm{Arg} \min_{l_i \leq x_i \leq u_i} \underbrace{\frac{L}{2} \left(x_i - [s_L(x^k)]_i \right)^2 + \lambda \|x_i\|_0}_{h(x_i)}.
$$

We divide the proof into several cases below.

case (1) : $[s_L(x^k)]_i = 0$. Since $l_i \leq 0$ and $u_i \geq 0$, one can see that $x_i^{k+1} = 0$ and the conclusion holds.

case (2) : $[s_L(x^k)]_i \neq 0$. We can observe that $[\Pi_{\mathcal{B}}(s_L(x^k))]_i \neq 0$ and moreover

$$
\min_{l_i \leq x_i \leq u_i, x_i \neq 0} h(x_i) = \frac{L}{2} [\Pi_{\mathcal{B}}(s_L(x^k)) - s_L(x^k)]_i^2 + \lambda,
$$

where the minimal value is achieved at $x_i = \Pi_{\mathcal{B}}(s_L(x^k))]_i \neq 0$. Notice that $h(0) = L[s_L(x^k)]_i^2/2$. Using these facts, it is not hard to see that the conclusion again holds.

∎

The following lemma shows that for the sequence $\{x^k\}$, the magnitude of any nonzero component x_i^k cannot be too small for $k \geq 1$.

Lemma 3.3 *Let $\{x^k\}$ be generated by the above IHT method. Then there hold:*
(i) *for all $k \geq 0$,*
$$
|x_j^{k+1}| \geq \delta := \min_{i \notin I_0} \delta_i > 0, \quad \text{if } x_j^{k+1} \neq 0, \tag{21}
$$

where $I_0 = \{i : l_i = u_i = 0\}$ and

$$
\delta_i = \begin{cases} \min(u_i, \sqrt{2\lambda/L}), & \text{if } l_i = 0, \\ \min(-l_i, \sqrt{2\lambda/L}), & \text{if } u_i = 0, \quad \forall i \notin I_0. \\ \min(-l_i, u_i, \sqrt{2\lambda/L}), & \text{otherwise,} \end{cases} \tag{22}
$$

(ii) *for all $k \geq 1$,*
$$
\|x^{k+1} - x^k\| \geq \delta \quad \text{if } I(x^k) \neq I(x^{k+1}). \tag{23}
$$

9

Proof. (i) Suppose that j is an index such that $x_j^{k+1} \neq 0$. Clearly, $j \notin I_0$, where I_0 is defined above. It follows from (20) that

$$x_j^{k+1} = [\Pi_{\mathcal{B}}(s_L(x^k))]_j \neq 0, \qquad [s_L(x^k)]_j^2 - [\Pi_{\mathcal{B}}(s_L(x^k)) - s_L(x^k)]_j^2 \geq \frac{2\lambda}{L}. \tag{24}$$

The second relation of (24) implies that $|[s_L(x^k)]_j| \geq \sqrt{2\lambda/L}$. In addition, by the first relation of (24) and the definition of $\Pi_{\mathcal{B}}$, we have

$$x_j^{k+1} = [\Pi_{\mathcal{B}}(s_L(x^k))]_j = \min(\max([s_L(x^k)]_j, l_j), u_j) \neq 0. \tag{25}$$

Recall that $j \notin I_0$. We next show that $|x_j^{k+1}| \geq \delta_j$ by considering three separate cases: i) $l_j = 0$; ii) $u_j = 0$; and iii) $l_j u_j \neq 0$. For case i), it follows from (25) that $[s_L(x^k)]_j \geq 0$ and $x_j^{k+1} = \min([s_L(x^k)]_j, u_j)$. This together with the relation $|[s_L(x^k)]_j| \geq \sqrt{2\lambda/L}$ and the definition of δ_j implies that $|x_j^{k+1}| \geq \delta_j$. By the similar arguments, we can show that $|x_j^{k+1}| \geq \delta_j$ also holds for the other two cases. It is then easy to see that statement (i) holds.

(ii) Suppose that $I(x^k) \neq I(x^{k+1})$ for some $k \geq 1$. Then there exists some i such that $(x_i^k \neq 0$ but $x_i^{k+1} = 0)$ or $(x_i^k = 0$ but $x_i^{k+1} \neq 0)$, which together with (i) implies that $|x_i^k| \geq \delta$ or $|x_i^{k+1}| \geq \delta$ and moreover

$$\|x^{k+1} - x^k\| \geq |x_i^{k+1} - x_i^k| = \max(|x_i^k|, |x_i^{k+1}|) \geq \delta.$$

∎

We next establish that the sequence $\{x^k\}$ converges to a local minimizer of (12), and moreover, $F(x^k)$ converges to a local minimum value of (12).

Theorem 3.4 *Let $\{x^k\}$ be generated by the above IHT method. Then there hold:*

(i) *$I(x^k)$ changes only finitely often, where $I(\cdot)$ is defined in (18).*

(ii) *$\{x^k\}$ converges to a local minimizer x^* of problem (12). Moreover, $I(x^k) \to I(x^*)$, $\|x^k\|_0 \to \|x^*\|_0$, $F(x^k) \to F(x^*)$, and*

$$x^* \in \operatorname{Arg\,min}\left\{f(x) : x \in \mathcal{B}_{I(x^*)}\right\}. \tag{26}$$

Proof. (i) Since ∇f is Lipschitz continuous with constant L_f, we have

$$f(x^{k+1}) \leq f(x^k) + \nabla f(x^k)^T(x^{k+1} - x^k) + \frac{L_f}{2}\|x^{k+1} - x^k\|^2.$$

Using this inequality, the fact that $L > L_f$, and (14), we obtain that

$$F(x^{k+1}) = f(x^{k+1}) + \lambda\|x^{k+1}\|_0 \leq \overbrace{f(x^k) + \nabla f(x^k)^T(x^{k+1} - x^k) + \frac{L_f}{2}\|x^{k+1} - x^k\|^2 + \lambda\|x^{k+1}\|_0}^{a},$$

$$\leq \underbrace{f(x^k) + \nabla f(x^k)^T(x^{k+1} - x^k) + \frac{L}{2}\|x^{k+1} - x^k\|^2 + \lambda\|x^{k+1}\|_0}_{b}$$

$$\leq f(x^k) + \lambda\|x^k\|_0 = F(x^k),$$

10

where the last inequality follows from (14). The above inequality implies that $\{F(x^k)\}$ is nonincreasing and moreover,

$$F(x^k) - F(x^{k+1}) \geq b - a = \frac{L - L_f}{2}\|x^{k+1} - x^k\|^2. \tag{27}$$

By the assumption, we know that f is bounded below in $\mathcal{B}$. It then follows that $\{F(x^k)\}$ is bounded below. Hence, $\{F(x^k)\}$ converges to a finite value as $k \to \infty$, which together with (27) implies that

$$\lim_{k \to \infty} \|x^{k+1} - x^k\| = 0,$$

which together with (23) implies that $I(x^k)$ does not change when k is sufficient large.

(ii) It follows from statement (i) that there exist some $K \geq 0$ and $I \subseteq \{1, \ldots, n\}$ such that $I(x^k) = I$ for all $k \geq K$. Then one can observe from (14) that

$$x^{k+1} = \arg\min_{x \in \mathcal{B}_I}\{f(x^k) + \nabla f(x^k)^T(x - x^k) + \frac{L}{2}\|x - x^k\|^2\}, \quad \forall k > K,$$

where $\mathcal{B}_I$ is defined in (15). It follows from Theorem 2.2 that $x^k \to x^*$, where

$$x^* \in \operatorname{Arg}\min\{f(x) : x \in \mathcal{B}_I\}. \tag{28}$$

It is not hard to see from (28) that x^* is a local minimizer of (12). In addition, we know from (21) that $|x_i^k| \geq \delta$ for $k > K$ and $i \notin I$. It yields $|x_i^*| \geq \delta$ for $i \notin I$ and $x_i^* = 0$ for $i \in I$. Hence, $I(x^k) = I(x^*) = I$ for all $k > K$, which clearly implies that $\|x^k\|_0 = \|x^*\|_0$ for every $k > K$. By continuity of f, we have $f(x^k) \to f(x^*)$. It then follows that

$$F(x^k) = f(x^k) + \lambda\|x^k\|_0 \to f(x^*) + \lambda\|x^*\|_0 = F(x^*).$$

Finally, (26) holds due to (28) and the relation $I(x^*) = I$. ∎

As shown in Theorem 3.4, $x^k \to x^*$ for some local minimizer x^* of (12) and $F(x^k) \to F(x^*)$. Our next aim is to establish the iteration complexity of the IHT method for finding an ϵ-local-optimal solution $x_\epsilon \in \mathcal{B}$ of (12) satisfying $F(x_\epsilon) \leq F(x^*) + \epsilon$ and $I(x_\epsilon) = I(x^*)$. Before proceeding, we define

$$\alpha = \min_{I \subseteq \{1,\ldots,n\}} \left\{ \min_i \left| [s_L(x^*)]_i^2 - [\Pi_{\mathcal{B}}(s_L(x^*)) - s_L(x^*)]_i^2 - \frac{2\lambda}{L} \right| : x^* \in \operatorname{Arg}\min\{f(x) : x \in \mathcal{B}_I\} \right\} \tag{29}$$

$$\beta = \max_{I \subseteq \{1,\ldots,n\}} \left\{ \max_i |[s_L(x^*)]_i| + |[\Pi_{\mathcal{B}}(s_L(x^*)) - s_L(x^*)]_i| : x^* \in \operatorname{Arg}\min\{f(x) : x \in \mathcal{B}_I\} \right\}. \tag{30}$$

Theorem 3.5 *Assume that f is a smooth strongly convex function with modulus $\sigma > 0$. Suppose that $L > L_f$ is chosen such that $\alpha > 0$.* [3] *Let $\{x^k\}$ be generated by the above IHT method, $I_k = I(x^k)$ for all k, $x^* = \lim_{k \to \infty} x^k$, and $F^* = F(x^*)$. Then, for any given $\epsilon > 0$, the following statements hold:*

---

[3]The existence of such L is proven in the appendix.

(i) *The number changes of I_k is at most $\left\lfloor \frac{2(F(x^0)-F^*)}{(L-L_f)\delta^2} \right\rfloor$.*

(ii) *The total number of iterations by the IHT method for finding an ϵ-local-optimal solution $x_\epsilon \in \mathcal{B}$ satisfying $I(x_\epsilon) = I(x^*)$ and $F(x_\epsilon) \le F^* + \epsilon$ is at most $2\lceil L/\sigma \rceil \log \frac{\theta}{\epsilon}$, where*

$$\theta = (F(x^0) - F^*)2^{\frac{\omega+3}{2}}, \quad \omega = \max_t \left\{ (d-2c)t - ct^2 : 0 \le t \le \left\lfloor \frac{2(F(x^0)-F^*)}{(L-L_f)\delta^2} \right\rfloor \right\}, \quad (31)$$

$$c = \frac{(L-L_f)\delta^2}{2(F(x^0)-\underline{F}^*)}, \qquad \gamma = \sigma(\sqrt{2\alpha + \beta^2} - \beta)^2/32, \tag{32}$$

$$d = 2\log(F(x^0) - \underline{F}^*) + 4 - 2\log\gamma + c.$$

Proof. (i) By Theorem 3.4 (i), we know that I_k only changes for a finite number of times. Assume that I_k only changes at $k = n_1 + 1, \ldots, n_J + 1$, that is,

$$I_{n_{j-1}+1} = \cdots = I_{n_j} \ne I_{n_j+1} = \cdots = I_{n_{j+1}}, \ j = 1, \ldots, J-1, \tag{33}$$

where $n_0 = 0$.

We next bound J, i.e., the total number of changes of I_k. In view of (23) and (33), one can observe that

$$\|x^{n_j+1} - x^{n_j}\| \ge \delta, \quad j = 1, \ldots, J,$$

which together with (27) implies that

$$F(x^{n_j}) - F(x^{n_j+1}) \ge \frac{1}{2}(L - L_f)\delta^2, \quad j = 1, \ldots, J. \tag{34}$$

Summing up these inequalities and using the monotonicity of $\{F(x^k)\}$, we have

$$\frac{1}{2}(L - L_f)\delta^2 J \ \le \ F(x^{n_1}) - F(x^{n_J+1}) \ \le \ F(x^0) - F^*, \tag{35}$$

and hence

$$J \ \le \ \left\lfloor \frac{2(F(x^0) - F^*)}{(L - L_f)\delta^2} \right\rfloor. \tag{36}$$

(ii) Let n_j be defined as above for $j = 1, \ldots, J$. We first show that

$$n_j - n_{j-1} \ \le \ 2 + 2\lceil L/\sigma \rceil \left\lceil \log \left(F(x^0) - (j-1)(L - L_f)\delta^2/2 - \underline{F}^* \right) - \log\gamma \right\rceil, \quad j = 1, \ldots, J, \tag{37}$$

where $\underline{F}^*$ and γ are defined in (12) and (32), respectively. Indeed, one can observe from (14) that

$$x^{k+1} = \arg\min_{x\in\mathcal{B}}\{f(x^k) + \nabla f(x^k)^T(x - x^k) + \frac{L}{2}\|x - x^k\|^2 : x_{I_{k+1}} = 0\}.$$

Therefore, for $j = 1, \ldots, J$ and $k = n_{j-1}, \ldots, n_j - 1$,

$$x^{k+1} = \arg\min_{x\in\mathcal{B}}\{f(x^k) + \nabla f(x^k)^T(x - x^k) + \frac{L}{2}\|x - x^k\|^2 : x_{I_{n_j}} = 0\}.$$

12

We arbitrarily choose $1 \leq j \leq J$. Let $\bar{x}^*$ (depending on j) denote the optimal solution of

$$\min_{x \in \mathcal{B}} \{ f(x) : x_{I_{n_j}} = 0 \}. \tag{38}$$

One can observe that

$$\|\bar{x}^*\|_0 \leq \|x^{n_{j-1}+1}\|_0.$$

Also, it follows from (34) and the monotonicity of $\{F(x^k)\}$ that

$$F(x^{n_j+1}) \leq F(x^0) - \frac{j}{2}(L - L_f)\delta^2, \quad j = 1, \dots, J. \tag{39}$$

Using these relations and the fact that $F(\bar{x}^*) \geq \underline{F}^*$, we have

$$
\begin{aligned}
f(x^{n_{j-1}+1}) - f(\bar{x}^*) &= F(x^{n_{j-1}+1}) - \lambda\|x^{n_{j-1}+1}\|_0 - F(\bar{x}^*) + \lambda\|\bar{x}^*\|_0, \\
&\leq F(x^0) - \frac{j-1}{2}(L - L_f)\delta^2 - \underline{F}^*.
\end{aligned}
\tag{40}
$$

Suppose for a contradiction that (37) does not hold for some $1 \leq j \leq J$. Hence, we have

$$n_j - n_{j-1} > 2 + 2\lceil L/\sigma \rceil \left\lceil \log \left(F(x^0) - (j-1)(L - L_f)\delta^2/2 - \underline{F}^* \right) - \log \gamma \right\rceil.$$

This inequality and (40) yields

$$n_j - n_{j-1} > 2 + 2\lceil L/\sigma \rceil \left\lceil \log \frac{f(x^{n_{j-1}+1}) - f(\bar{x}^*)}{\gamma} \right\rceil.$$

Using the strong convexity of f and applying Theorem 2.3 (ii) to (38) with $\epsilon = \gamma$, we obtain that

$$\frac{\sigma}{2}\|x^{n_j} - \bar{x}^*\|^2 \leq f(x^{n_j}) - f(\bar{x}^*) < \frac{\sigma}{32}(\sqrt{2\alpha + \beta^2} - \beta)^2.$$

It implies that

$$\|x^{n_j} - \bar{x}^*\| < \frac{\sqrt{2\alpha + \beta^2} - \beta}{4}. \tag{41}$$

Using (41), Lemma 3.1 and the definition of β, we have

$$
\begin{aligned}
&|[s_L(x^{n_j})]_i^2 - [s_L(\bar{x}^*)]_i^2 - [\Pi_\mathcal{B}(s_L(x^{n_j})) - s_L(x^{n_j})]_i^2 + [\Pi_\mathcal{B}(s_L(\bar{x}^*)) - s_L(\bar{x}^*)]_i^2| \\
&\leq |[s_L(x^{n_j})]_i^2 - [s_L(\bar{x}^*)]_i^2| + |[\Pi_\mathcal{B}(s_L(x^{n_j})) - s_L(x^{n_j})]_i^2 - [\Pi_\mathcal{B}(s_L(\bar{x}^*)) - s_L(\bar{x}^*)]_i^2| \\
&\leq 4(2\|x^{n_j} - \bar{x}^*\| + \beta)\|x^{n_j} - \bar{x}^*\| < \alpha,
\end{aligned}
\tag{42}
$$

where the last inequality is due to (41). Let

$$I^* = \left\{ i : [s_L(\bar{x}^*)]_i^2 - [\Pi_\mathcal{B}(s_L(\bar{x}^*)) - s_L(\bar{x}^*)]_i^2 < \frac{2\lambda}{L} \right\}$$

and let $\bar{I}^* = \{1, \ldots, n\} \setminus I^*$. Since $\alpha > 0$, we know that

$$[s_L(\bar{x}^*)]_i^2 - [\Pi_{\mathcal{B}}(s_L(\bar{x}^*)) - s_L(\bar{x}^*)]_i^2 > \frac{2\lambda}{L}, \quad \forall i \in \bar{I}^*.$$

It then follows from (42) and the definition of α that

$$[s_L(x^{n_j})]_i^2 - [\Pi_{\mathcal{B}}(s_L(x^{n_j})) - s_L(x^{n_j})]_i^2 < \tfrac{2\lambda}{L}, \quad \forall i \in I^*,$$
$$[s_L(x^{n_j})]_i^2 - [\Pi_{\mathcal{B}}(s_L(x^{n_j})) - s_L(x^{n_j})]_i^2 > \tfrac{2\lambda}{L}, \quad \forall i \in \bar{I}^*.$$

Observe that $[\Pi_{\mathcal{B}}(s_L(x^{n_j}))]_i \neq 0$ for all $i \in \bar{I}^*$. This fact together with (20) implies that

$$x_i^{n_j+1} = 0, \ i \in I^* \ \text{ and } \ x_i^{n_j+1} \neq 0, \ i \in \bar{I}^*.$$

By a similar argument, one can show that

$$x_i^{n_j} = 0, \ i \in I^* \ \text{ and } \ x_i^{n_j} \neq 0, \ i \in \bar{I}^*.$$

Hence, $I_{n_j} = I_{n_j+1} = I^*$, which is a contradiction to (33). We thus conclude that (37) holds.

Let N_ϵ denote the total number of iterations for finding an ϵ-local-optimal solution $x_\epsilon \in \mathcal{B}$ by the IHT method satisfying $I(x_\epsilon) = I(x^*)$ and $F(x_\epsilon) \leq F^* + \epsilon$. We next establish an upper bound for N_ϵ. Summing up the inequality (37) for $j = 1, \ldots, J$, we obtain that

$$n_J \leq \sum_{j=1}^{J} \left\{ 2 + 2\lceil L/\sigma \rceil \left\lceil \log(F(x^0) - \frac{j-1}{2}(L - L_f)\delta^2 - \underline{F}^*) - \log\gamma \right\rceil \right\}.$$

Using this inequality, (35), and the facts that $L \geq \sigma$ and $\log(1 - t) \leq -t$ for all $t \in (0, 1)$, we have

$$n_J \leq \sum_{j=1}^{J} \left[2 + 2\lceil L/\sigma \rceil \left(\log(F(x^0) - \frac{j-1}{2}(L - L_f)\delta^2 - \underline{F}^*) - \log\gamma + 1 \right) \right],$$

$$\leq \sum_{j=1}^{J} \left[2 + 2\lceil L/\sigma \rceil \left(\log(F(x^0) - \underline{F}^*) - \frac{(L - L_f)\delta^2}{2(F(x^0) - \underline{F}^*)}(j - 1) - \log\gamma + 1 \right) \right],$$

$$\leq \lceil L/\sigma \rceil \left[\underbrace{\left(2\log(F(x^0) - \underline{F}^*) + 4 - 2\log\gamma + \frac{(L - L_f)\delta^2}{2(F(x^0) - \underline{F}^*)} \right)}_{d} J - \underbrace{\frac{(L - L_f)\delta^2}{2(F(x^0) - \underline{F}^*)}}_{c} J^2 \right] \tag{43}$$

By the definition of n_J, we observe that after $n_J + 1$ iterations, the IHT method becomes the projected gradient method applied to the problem

$$x^* = \arg\min_{x \in \mathcal{B}} \{ f(x) : x_{I_{n_J+1}} = 0 \}.$$

In addition, combining Theorem 3.4 (i) and (ii) and using the definition of n_J, one can see that $I(x^k) = I(x^*)$ for all $k > n_J$. Hence, $f(x^k) - f(x^*) = F(x^k) - F^*$ when $k > n_J$. Using these facts and Theorem 2.3 (ii), we have

$$N_\epsilon \leq n_J + 1 + 2\lceil L/\sigma \rceil \left\lceil \log \frac{F(x^{n_J+1}) - F^*}{\epsilon} \right\rceil.$$

Using this inequality, (39), (43) and the facts that $F^* \geq \underline{F}^*$, $L \geq \sigma$ and $\log(1 - t) \leq -t$ for all $t \in (0, 1)$, we obtain that

$$
\begin{aligned}
N_\epsilon \;\leq\;& n_J + 1 + 2\lceil L/\sigma \rceil \left(\log(F(x^0) - \frac{J}{2}(L - L_f)\delta^2 - F^*) + 1 - \log \epsilon \right), \\
\leq\;& n_J + \lceil L/\sigma \rceil \left(2\log(F(x^0) - F^*) - \frac{(L - L_f)\delta^2 J}{F(x^0) - F^*} + 3 - 2\log \epsilon \right) \\
\leq\;& \lceil L/\sigma \rceil \left[(d - 2c)J - cJ^2 + 2\log(F(x^0) - F^*) + 3 - 2\log \epsilon \right],
\end{aligned}
$$

which together with (36) and (31) implies that

$$N_\epsilon \;\leq\; 2\lceil L/\sigma \rceil \log \frac{\theta}{\epsilon}.$$

∎

The iteration complexity given in Theorem 3.5 is based on the assumption that f is strongly convex in $\mathcal{B}$. We next consider a case where $\mathcal{B}$ is bounded and f is convex but not strongly convex. We will establish the iteration complexity of finding an ϵ-local-optimal solution of (12) by the IHT method applied to a perturbation of (12) obtained from adding a "small" strongly convex regularization term to f.

Consider a perturbation of (12) in the form of

$$\underline{F}^*_\nu := \min_{x \in \mathcal{B}} \{ F_\nu(x) := f_\nu(x) + \lambda \|x\|_0 \}, \tag{44}$$

where $\nu > 0$ and

$$f_\nu(x) := f(x) + \frac{\nu}{2} \|x\|^2.$$

One can easily see that f_ν is strongly convex in $\mathcal{B}$ with modulus ν and moreover ∇f_ν is Lipschitz continuous with constant L_ν, where

$$L_\nu = L_f + \nu. \tag{45}$$

We next establish the iteration complexity of finding an ϵ-local-optimal solution of (12) by the IHT method applied to (44). Given any $L > 0$, let s_L, α and β be defined according to (17), (29) and (30), respectively, by replacing f by f_ν, and let δ be defined in (21).

15

Theorem 3.6 *Suppose that $\mathcal{B}$ is bounded and f is convex but not strongly convex. Let $\epsilon > 0$ be arbitrarily given, $D = \max\{\|x\| : x \in \mathcal{B}\}$, $\nu = \epsilon/D^2$, and $L > L_\nu$ be chosen such that $\alpha > 0$. Let $\{x^k\}$ be generated by the IHT method applied to (44), and let $x^* = \lim_{k \to \infty} x^k$, $F_\nu^* = F_\nu(x^*)$ and $F^* = \min\{F(x) : x \in \mathcal{B}_{I^*}\}$, where $I^* = \{i : x_i^* = 0\}$. Then, the total number of iterations by the IHT method for finding an ϵ-local-optimal solution $x_\epsilon \in \mathcal{B}$ satisfying $F(x_\epsilon) \leq F^* + \epsilon$ is at most $2\left\lceil \frac{D^2 L_f}{\epsilon} + 1 \right\rceil \log \frac{2\theta}{\epsilon}$, where*

$$\theta = (F_\nu(x^0) - F_\nu^*)2^{\frac{\omega+3}{2}}, \quad \omega = \max_t \left\{ (d - 2c)t - ct^2 : 0 \leq t \leq \left\lfloor \frac{2(F_\nu(x^0) - F_\nu^*)}{(L - L_\nu)\delta^2} \right\rfloor \right\},$$

$$c = \frac{(L - L_\nu)\delta^2}{2(F_\nu(x^0) - \underline{F}_\nu^*)}, \qquad \gamma = \nu(\sqrt{2\alpha + \beta^2} - \beta)^2/32,$$

$$d = 2\log(F_\nu(x^0) - \underline{F}_\nu^*) + 4 - 2\log\gamma + c.$$

Proof. By Theorem 3.5 (ii), we see that the IHT method applied to (44) finds an $\epsilon/2$-local-optimal solution $x_\epsilon \in \mathcal{B}$ of (44) satisfying $I(x_\epsilon) = I(x^*)$ and $F_\nu(x_\epsilon) \leq F_\nu^* + \epsilon/2$ within $2\lceil L_\nu/\nu \rceil \log \frac{2\theta}{\epsilon}$ iterations. Also, we know from Theorem 3.4 (ii) that

$$F_\nu(x^*) = \min\{F_\nu(x) : x \in \mathcal{B}_{I^*}\}.$$

Hence, we have

$$F_\nu^* \;=\; F_\nu(x^*) \;\leq\; \min_{x \in \mathcal{B}_{I^*}} f(x) + \frac{\nu D^2}{2} \;\leq\; F^* + \frac{\epsilon}{2}.$$

In addition, we observe that $F(x_\epsilon) \leq F_\nu(x_\epsilon)$. Hence, it follows that

$$F(x_\epsilon) \;\leq\; F_\nu(x_\epsilon) \;\leq\; F_\nu^* + \frac{\epsilon}{2} \;\leq\; F^* + \epsilon.$$

Note that F^* is a local optimal value of (12). Hence, x_ϵ is an ϵ-local-optimal solution of (12). The conclusion of this theorem then follows from (45) and $\nu = \epsilon/D^2$. ■

For the above IHT method, a fixed L is used through all iterations, which may be too conservative. To improve its practical performance, we can use a "local" L that is updated dynamically. The resulting variant of the method is presented as follows. Several approaches have been proposed to address this issue for solving the l_0-constrained least squares problems in the context of compressed sensing (see, for example, [17, 7, 9, 12, 3]).

A variant of IHT method for (12):

Let $0 < L_{\min} < L_{\max}$, $\tau > 1$ and $\eta > 0$ be given. Choose an arbitrary $x^0 \in \mathcal{B}$ and set $k = 0$.

1) Choose $L_k^0 \in [L_{\min}, L_{\max}]$ arbitrarily. Set $L_k = L_k^0$.

   1a) Solve the subproblem

$$x^{k+1} \in \operatorname*{Arg\,min}_{x \in \mathcal{B}}\{f(x^k) + \nabla f(x^k)^T(x - x^k) + \frac{L_k}{2}\|x - x^k\|^2 + \lambda\|x\|_0\}. \tag{46}$$

16

1b) If

$$F(x^k) - F(x^{k+1}) \geq \frac{\eta}{2}\|x^{k+1} - x^k\|^2 \tag{47}$$

is satisfied, then go to step 2).

1c) Set $L_k \leftarrow \tau L_k$ and go to step 1a).

2) Set $k \leftarrow k + 1$ and go to step 1).

end

Remark. In the above variant of IHT method, the Armijo-type backtracking procedure is used to find a suitable L^k starting with L_k^0, which can be chosen as the one proposed by Barzilai and Borwein [2], that is,

$$L_k^0 = \max\left\{L_{\min}, \min\left\{L_{\max}, \frac{\Delta f^T \Delta x}{\|\Delta x\|^2}\right\}\right\},$$

where $\Delta x = x^k - x^{k-1}$ and $\Delta f = \nabla f(x^k) - \nabla f(x^{k-1})$. ∎

At each iteration, the IHT method solves a single subproblem in step 1). Nevertheless, its variant needs to solve a sequence of subproblems. We next show that for each outer iteration, its number of inner iterations is finite.

Theorem 3.7 *For each $k \geq 0$, the inner termination criterion* (47) *is satisfied after at most* $\left\lceil \frac{\log(L_f+\eta)-\log(L_{\min})}{\log \tau} + 2 \right\rceil$ *inner iterations.*

Proof. Let $\bar{L}_k$ denote the final value of L_k at the kth outer iteration. By (46) and the similar arguments as for deriving (27), one can show that

$$F(x^k) - F(x^{k+1}) \geq \frac{L_k - L_f}{2}\|x^{k+1} - x^k\|^2.$$

Hence, (47) holds whenever $L_k \geq L_f + \eta$, which together with the definition of $\bar{L}_k$ implies that $\bar{L}_k/\tau < L_f + \eta$, that is, $\bar{L}_k < \tau(L_f + \eta)$. Let n_k denote the number of inner iterations for the kth outer iteration. Then, we have

$$L_{\min}\tau^{n_k-1} \leq L_k^0 \tau^{n_k-1} = \bar{L}_k < \tau(L_f + \eta).$$

Hence, $n_k \leq \left\lceil \frac{\log(L_f+\eta)-\log(L_{\min})}{\log \tau} + 2 \right\rceil$ and the conclusion holds. ∎

We next establish that the sequence $\{x^k\}$ generated by the above variant of IHT method converges to a local minimizer of (12) and moreover, $F(x^k)$ converges to a local minimum value of (12).

17

Theorem 3.8 *Let $\{x^k\}$ be generated by the above variant of IHT method. Then, x^k converges to a local minimizer x^* of problem (12). Moreover, $I(x^k) \to I(x^*)$, $\|x^k\|_0 \to \|x^*\|_0$, $F(x^k) \to F(x^*)$, and*

$$x^* \in \text{Arg min}\left\{f(x) : x \in \mathcal{B}_{I(x^*)}\right\}.$$

Proof. Let $\bar{L}_k$ denote the final value of L_k at the kth outer iteration. From the proof of Theorem 3.7, we know that $\bar{L}_k \in [L_{\min}, \tau(L_f + \eta))$. Using this fact and a similar argument as used to prove (21), one can obtain that

$$|x_i^{k+1}| \geq \bar{\delta} := \min_{i \notin I_0} \bar{\delta}_i > 0, \quad \text{if } x_j^{k+1} \neq 0,$$

where $I_0 = \{i : l_i = u_i = 0\}$ and $\bar{\delta}_i$ is defined according to (22) by replacing L by $\tau(L_f + \eta)$ for all $i \in I_0$. It implies that for $k \geq 1$,

$$\|x^{k+1} - x^k\| \geq \bar{\delta} \text{ if } I(x^k) \neq I(x^{k+1}).$$

The conclusion then follows from this inequality and the similar arguments as used in the proof of Theorem 3.4. ∎

4 l_0-regularized convex cone programming

In this section we consider l_0-regularized convex cone programming problem (2) and propose IHT methods for solving it. In particular, we apply the IHT method proposed in Section 3 to a quadratic penalty relaxation of (2) and establish the iteration complexity for finding an ϵ-approximate local minimizer of (2). We also propose a variant of the method in which the associated penalty parameter is dynamically updated, and show that every accumulation point is a local minimizer of (2).

Let $\mathcal{B}$ be defined in (13). We assume that f is a smooth convex function in $\mathcal{B}$, ∇f is Lipschitz continuous with constant L_f and that f is bounded below on $\mathcal{B}$. In addition, we make the following assumption throughout this section.

Assumption 1 *For each $I \subseteq \{1, \ldots, n\}$, there exists a Lagrange multiplier for*

$$f_I^* = \min\{f(x) : Ax - b \in \mathcal{K}^*, x \in \mathcal{B}_I\}, \tag{48}$$

provided that (48) is feasible, that is, there exists $\mu^* \in -\mathcal{K}$ such that $f_I^* = d_I(\mu^*)$, where

$$d_I(\mu) := \inf\{f(x) + \mu^T(Ax - b) : x \in \mathcal{B}_I\}, \ \forall \mu \in -\mathcal{K}.$$

Remark. Problem (48) may be infeasible for some $I \subseteq \{1, \ldots, n\}$. For example, if $-b \notin \mathcal{K}^*$, then (48) is infeasible for $I = \{1, \ldots, n\}$ since $\mathcal{B}_I = \{0\}$. As mentioned below, any local minimizer of (2) is a minimizer of (48) for some I. Since our aim is to solve (2), we are thus only interested in those $I \subseteq \{1, \ldots, n\}$ for which (48) is feasible. As we will later see, the IHT

18

methods proposed for solving (2) become at their late stage some first-order methods applied to (48) for some I. In addition, it is common to assume the strong duality holds when solving a convex cone programming (see, for example, [15]). Assumption 1 is thus very reasonable. ∎

Let x^* be a point in $\mathcal{B}$, and let $I^* = \{i : x_i^* = 0\}$. One can observe that x^* is a local minimizer of (2) if and only if x^* is a minimizer of (48) with $I = I^*$. Then, in view of Assumption 1, we see that x^* is a local minimizer of (2) if and only if $x^* \in \mathcal{B}$ and there exists $\mu^* \in -\mathcal{K}$ such that

$$
\begin{aligned}
Ax^* - b \in \mathcal{K}^*, \quad (\mu^*)^T (Ax^* - b) &= 0, \\
\nabla f(x^*) + A^T \mu^* &\in -\mathcal{N}_{\mathcal{B}_{I^*}}(x^*).
\end{aligned}
\tag{49}
$$

Based on the above observation, we can define an approximate local minimizer of (2) to be the one that nearly satisfies (49).

Definition 1 *Let x^* be a point in $\mathcal{B}$, and let $I^* = \{i : x_i^* = 0\}$. x^* is an ϵ-approximate local minimizer of* (2) *if there exists $\mu^* \in -\mathcal{K}$ such that*

$$
\begin{aligned}
d_{\mathcal{K}^*}(Ax^* - b) \leq \epsilon, \quad (\mu^*)^T \Pi_{\mathcal{K}^*}(Ax^* - b) &= 0, \\
\nabla f(x^*) + A^T \mu^* &\in -\mathcal{N}_{\mathcal{B}_{I^*}}(x^*) + \mathcal{U}(\epsilon).
\end{aligned}
$$

In what follows, we propose an IHT method for finding an approximate local minimizer of (2). In particular, we apply the IHT method or its variant to a quadratic penalty relaxation of (2) which is in the form of

$$
\underline{\Psi}_\rho^* := \min_{x \in \mathcal{B}} \{\Psi_\rho(x) := \Phi_\rho(x) + \lambda \|x\|_0\},
\tag{50}
$$

where

$$
\Phi_\rho(x) := f(x) + \frac{\rho}{2}[d_{\mathcal{K}^*}(Ax - b)]^2
\tag{51}
$$

It is not hard to show that the function Φ_ρ is convex differentiable and moreover $\nabla \Phi_\rho$ is Lipschitz continuous with constant

$$
L_\rho = L_f + \rho \|A\|^2
\tag{52}
$$

(see, for example, Proposition 8 and Corollary 9 of [15]). Therefore, problem (50) can be suitably solved by the IHT method or its variant proposed in Section 3.

Under the assumption that f is strongly convex in $\mathcal{B}$, we next establish the iteration complexity of the IHT method applied to (50) for finding an approximate local minimizer of (2). Given any $L > 0$, let s_L, α and β be defined according to (17), (29) and (30), respectively, by replacing f by Φ_ρ, and let δ be defined in (21).

Theorem 4.1 *Assume that f is a smooth strongly convex function with modulus $\sigma > 0$. Given any $\epsilon > 0$, let*

$$
\rho = \frac{t}{\epsilon} + \frac{1}{\sqrt{8}\|A\|}
\tag{53}
$$

19

for any $t \geq \max\limits_{I \subseteq \{1,\ldots,n\}} \min\limits_{\mu \in \Lambda_I} \|\mu\|$, *where* Λ_I *is the set of Lagrange multipliers of* (48). *Let* $L > L_\rho$ *be chosen such that* $\alpha > 0$. *Let* $\{x^k\}$ *be generated by the IHT method applied to* (50), *and let* $x^* = \lim_{k\to\infty} x^k$ *and* $\Psi^*_\rho = \Psi_\rho(x^*)$. *Then the IHT method finds an* ϵ-*approximate local minimizer of* (2) *in at most*

$$N := 2 \left\lceil \frac{L_\rho}{\sigma} \right\rceil \log \frac{8 L_\rho \theta}{\epsilon^2}$$

iterations, where

$$\theta = (\Psi_\rho(x^0) - \Psi^*_\rho) 2^{\frac{\omega+3}{2}}, \quad \omega = \max_t \left\{ (d - 2c)t - ct^2 : 0 \leq t \leq \left\lfloor \frac{2(\Psi_\rho(x^0) - \Psi^*_\rho)}{(L - L_\rho)\delta^2} \right\rfloor \right\},$$

$$c = \frac{(L - L_\rho)\delta^2}{2(\Psi_\rho(x^0) - \underline{\Psi}^*_\rho)}, \qquad \gamma = \sigma(\sqrt{2\alpha + \beta^2} - \beta)^2/32,$$

$$d = 2\log(\Psi_\rho(x^0) - \underline{\Psi}^*_\rho) + 4 - 2\log\gamma + c.$$

Proof. We know from Theorem 3.4 (ii) that $x^k \to x^*$ for some local minimizer x^* of (50), $I(x^k) \to I(x^*)$, $\Psi_\rho(x^k) \to \Psi_\rho(x^*) = \Psi^*_\rho$, and

$$x^* = \arg\min_{x \in \mathcal{B}_{I^*}} \Phi_\rho(x), \tag{54}$$

where $I^* = I(x^*)$. By Theorem 3.5, after at most N iterations, the IHT method generates $\tilde{x} \in \mathcal{B}$ such at $I(\tilde{x}) = I(x^*)$ and $\Psi_\rho(\tilde{x}) - \Psi_\rho(x^*) \leq \xi := \epsilon^2/(8L_\rho)$. It then follows that $\Phi_\rho(\tilde{x}) - \Phi_\rho(x^*) \leq \xi$. Hence, $\tilde{x}$ is a ξ-approximate solution of (54). Let $\mu^* \in \text{Arg}\min\{\|\mu\| : \mu \in \Lambda_{I^*}\}$, where Λ_{I^*} is the set of Lagrange multipliers of (48) with $I = I^*$. In view of Lemma 2.5, we see that the pair $(\tilde{x}^+, \mu)$ defined as $\tilde{x}^+ := \Pi_{\mathcal{B}_{I^*}}(\tilde{x} - \nabla\Phi_\rho(\tilde{x})/L_\rho)$ and $\mu := \rho[A\tilde{x}^+ - b - \Pi_{\mathcal{K}^*}(A\tilde{x}^+ - b)]$ satisfies

$$\nabla f(\tilde{x}^+) + A^T\mu \in -\mathcal{N}_{\mathcal{B}_{I^*}}(\tilde{x}^+) + \mathcal{U}(2\sqrt{2L_\rho\xi}) = \mathcal{N}_{\mathcal{B}_I}(\tilde{x}^+) + \mathcal{U}(\epsilon),$$

$$d_{\mathcal{K}^*}(A\tilde{x}^+ - b) \leq \tfrac{1}{\rho}\|\mu^*\| + \sqrt{\tfrac{\xi}{\rho}} \leq \tfrac{1}{\rho}\left(\|\mu^*\| + \tfrac{\epsilon}{\sqrt{8}\|A\|}\right) \leq \epsilon,$$

where the last inequality is due to (53) and the assumption $t \geq \hat{t} \geq \|\mu^*\|$. Hence, $\tilde{x}^+$ is an ϵ-approximate local minimizer of (2). $\blacksquare$

We next consider finding an ϵ-approximate local minimizer of (2) for the case where $\mathcal{B}$ is bounded and f is convex but not strongly convex. In particular, we apply the IHT method or its variant to a quadratic penalty relaxation of a perturbation of (2) obtained from adding a "small' ' strongly convex regularization term to f.

Consider a perturbation of (2) in the form of

$$\min_{x \in \mathcal{B}} \{ f(x) + \frac{\nu}{2}\|x\|^2 + \lambda\|x\|_0 : \ Ax - b \in \mathcal{K}^* \}. \tag{55}$$

The associated quadratic penalty problem for (55) is given by

$$\underline{\Psi}^*_{\rho,\nu} := \min_{x \in \mathcal{B}} \{ \Psi_{\rho,\nu}(x) := \Phi_{\rho,\nu}(x) + \lambda\|x\|_0 \}, \tag{56}$$

20

where

$$\Phi_{\rho,\nu}(x) := f(x) + \frac{\nu}{2}\|x\|^2 + \frac{\rho}{2}[d_{\mathcal{K}^*}(Ax - b)]^2.$$

One can easily see that $\Phi_{\rho,\nu}$ is strongly convex in $\mathcal{B}$ with modulus ν and moreover $\nabla\Phi_{\rho,\nu}$ is Lipschitz continuous with constant

$$L_{\rho,\nu} := L_f + \rho\|A\|^2 + \nu.$$

Clearly, the IHT method or its variant can be suitably applied to (56). We next establish the iteration complexity of the IHT method applied to (56) for finding an approximate local minimizer of (2). Given any $L > 0$, let s_L, α and β be defined according to (17), (29) and (30), respectively, by replacing f by $\Phi_{\rho,\nu}$, and let δ be defined in (21).

Theorem 4.2 *Suppose that $\mathcal{B}$ is bounded and f is convex but not strongly convex. Let $\epsilon > 0$ be arbitrarily given, $D = \max\{\|x\| : x \in \mathcal{B}\}$,*

$$\rho = \frac{\left(\sqrt{D} + \sqrt{D + 16t + \frac{2\sqrt{2}\epsilon}{\|A\|}}\right)^2}{16\epsilon}, \qquad \nu = \frac{\epsilon}{2D} \tag{57}$$

for any $t \geq \max\limits_{I \subseteq \{1,\ldots,n\}} \min\limits_{\mu \in \Lambda_I} \|\mu\|$, where Λ_I is the set of Lagrange multipliers of (48). Let $L > L_{\rho,\nu}$ be chosen such that $\alpha > 0$. Let $\{x^k\}$ be generated by the IHT method applied to (56), and let $x^* = \lim_{k\to\infty} x^k$ and $\Psi^*_{\rho,\nu} = \Psi_{\rho,\nu}(x^*)$. Then the IHT method finds an ϵ-approximate local minimizer of (2) in at most

$$N := 2\left\lceil \frac{2DL_{\rho,\nu}}{\epsilon} \right\rceil \log \frac{32L_{\rho,\nu}\theta}{\epsilon^2}$$

iterations, where

$$\theta = (\Psi_{\rho,\nu}(x^0) - \Psi^*_{\rho,\nu})2^{\frac{\omega+3}{2}}, \qquad \omega = \max_t\left\{(d - 2c)t - ct^2 : 0 \leq t \leq \left\lfloor \frac{2(\Psi_{\rho,\nu}(x^0) - \Psi^*_{\rho,\nu})}{(L - L_{\rho,\nu})\delta^2} \right\rfloor\right\},$$

$$c = \frac{(L - L_{\rho,\nu})\delta^2}{2(\Psi_{\rho,\nu}(x^0) - \underline{\Psi}^*_{\rho,\nu})}, \qquad \gamma = \nu(\sqrt{2\alpha_{\rho,\nu} + \beta^2_{\rho,\nu}} - \beta_{\rho,\nu})^2/32,$$

$$d = 2\log(\Psi_{\rho,\nu}(x^0) - \underline{\Psi}^*_{\rho,\nu}) + 4 - 2\log\gamma + c.$$

Proof. From Theorem 3.4 (ii), we know that $x^k \to x^*$ for some local minimizer x^* of (56), $I(x^k) \to I(x^*)$, $\Psi_{\rho,\nu}(x^k) \to \Psi_{\rho,\nu}(x^*) = \Psi^*_{\rho,\nu}$, and

$$x^* = \arg\min_{x \in \mathcal{B}_{I^*}} \Phi_{\rho,\nu}(x), \tag{58}$$

where $I^* = I(x^*)$. By Theorem 3.5, after at most N iterations, the IHT method applied to (56) generates $\tilde{x} \in \mathcal{B}$ such at $I(\tilde{x}) = I(x^*)$ and $\Psi_{\rho,\nu}(\tilde{x}) - \Psi_{\rho,\nu}(x^*) \leq \xi := \epsilon^2/(32L_{\rho,\nu})$. It then follows that $\Phi_{\rho,\nu}(\tilde{x}) - \Phi_{\rho,\nu}(x^*) \leq \xi$. Hence, $\tilde{x}$ is a ξ-approximate solution of (58). In

view of Lemma 2.5, we see that the pair $(\tilde{x}^+, \mu)$ defined as $\tilde{x}^+ := \Pi_{\mathcal{B}_I}(\tilde{x} - \nabla\Phi_{\rho,\nu}(\tilde{x})/L_{\rho,\nu})$ and $\mu := \rho[A\tilde{x}^+ - b - \Pi_{\mathcal{K}^*}(A\tilde{x}^+ - b)]$ satisfies

$$\nabla f(\tilde{x}^+) + \nu\tilde{x}^+ + A^T\mu \in -\mathcal{N}_{\mathcal{B}_{I^*}}(\tilde{x}^+) + \mathcal{U}(2\sqrt{2L_{\rho,\nu}\xi}) = -\mathcal{N}_{\mathcal{B}_{I^*}}(\tilde{x}^+) + \mathcal{U}(\epsilon/2),$$

which together with the fact that $\nu\|\tilde{x}^+\| \leq \nu D \leq \epsilon/2$ implies that

$$\nabla f(\tilde{x}^+) + A^T\mu \in -\nu\tilde{x}^+ - \mathcal{N}_{\mathcal{B}_{I^*}}(\tilde{x}^+) + \mathcal{U}(\epsilon/2) \subseteq -\mathcal{N}_{\mathcal{B}_{I^*}}(\tilde{x}^+) + \mathcal{U}(\epsilon).$$

In addition, it follows from Lemma 2.1 (c) that $\Phi_{\rho,\nu}(\tilde{x}^+) \leq \Phi_{\rho,\nu}(\tilde{x})$, and hence

$$\Phi_{\rho,\nu}(\tilde{x}^+) - \Phi_{\rho,\nu}(x^*) \leq \Phi_{\rho,\nu}(\tilde{x}) - \Phi_{\rho,\nu}(x^*) \leq \xi.$$

Let $\Phi_\rho^* = \min\{\Phi_\rho(x) : x \in \mathcal{B}_{I^*}\}$, where Φ_ρ is defined in (51). Notice that $\Phi_{\rho,\nu}(x^*) \leq \Phi_\rho^* + \nu D^2/2$. It then follows that

$$\Phi_\rho(\tilde{x}^+) - \Phi_\rho^* \leq \Phi_{\rho,\nu}(\tilde{x}^+) - \Phi_{\rho,\nu}(x^*) + \frac{\nu D^2}{2} \leq \xi + \frac{\epsilon D}{4} \leq \frac{\epsilon^2}{32\rho\|A\|^2} + \frac{\epsilon D}{4}.$$

Let $\mu^* \in \text{Arg}\min\{\|\mu\| : \mu \in \Lambda_{I^*}\}$, where Λ_{I^*} is the set of Lagrange multipliers of (48) with $I = I^*$. In view of Lemma 2.5 and the assumption $t \geq \hat{t} \geq \|\mu^*\|$, we obtain that

$$d_{\mathcal{K}^*}(A\tilde{x}^+ - b) \leq \frac{1}{\rho}\|\mu^*\| + \sqrt{\frac{\epsilon^2}{32\rho^2\|A\|^2} + \frac{\epsilon D}{4\rho}} \leq \frac{1}{\rho}\left(t + \frac{\epsilon}{\sqrt{32}\|A\|}\right) + \sqrt{\frac{\epsilon D}{4\rho}} = \epsilon,$$

where the last inequality is due to (57). Hence, $\tilde{x}^+$ is an ϵ-approximate local minimizer of (2). ∎

For the above method, the fixed penalty parameter ρ is used through all iterations, which may be too conservative. To improve its practical performance, we can update ρ dynamically. The resulting variant of the method is presented as follows. Before proceeding, we define the projected gradient of Φ_ρ at $x \in \mathcal{B}_I$ with respect to $\mathcal{B}_I$ as

$$g(x; \rho, I) = L_\rho[x - \Pi_{\mathcal{B}_I}(x - \frac{1}{L_\rho}\nabla\Phi_\rho(x))], \tag{59}$$

where $I \subseteq \{1, \ldots, n\}$, and Φ_ρ and L_ρ are defined in (51) and (52), respectively.

A variant of IHT method for (2):

Let $\{\epsilon_k\}$ be a positive decreasing sequence. Let $\rho_0 > 0$, $\tau > 1$, $t > \max_{I \subseteq \{1,\ldots,n\}} \min_{\mu \in \Lambda_I} \|\mu\|$, where Λ_I is the set of Lagrange multipliers of (48). Choose an arbitrary $x^0 \in \mathcal{B}$. Set $k = 0$.

1) Start from x^{k-1} and apply the IHT method or its variant to problem (50) with $\rho = \rho_k$ until finding some $x^k \in \mathcal{B}$ such that

$$d_{\mathcal{K}^*}(Ax^k - b) \leq \frac{t}{\rho_k}, \qquad \|g(x^k; \rho_k, I_k)\| \leq \min\{1, L_{\rho_k}\}\epsilon_k, \tag{60}$$

where $I_k = I(x^k)$.

2) Set $\rho_{k+1} := \tau\rho_k$.

3) Set $k \leftarrow k + 1$ and go to step 1).

end

The following theorem shows that x^k satisfying (60) can be found within a finite number of iterations by the IHT method or its variant applied to problem (50) with $\rho = \rho_k$. Without loss of generality, we consider the IHT method or its variant applied to problem (50) with any given $\rho > 0$.

Theorem 4.3 *Let $x_0 \in \mathcal{B}$ be an arbitrary point and the sequence $\{x_l\}$ be generated by the IHT method or its variant applied to problem (50). Then, the following statements hold:*

(i) $\lim\limits_{l \to \infty} g(x_l; \rho, I_l) = 0$, *where $I_l = I(x_l)$ for all l.*

(ii) $\lim\limits_{l \to \infty} d_{\mathcal{K}^*}(Ax_l - b) \le \frac{\hat{t}}{\rho}$, *where $\hat{t} := \max\limits_{I \subseteq \{1,\dots,n\}} \min\limits_{\mu \in \Lambda_I} \|\mu\|$ and Λ_I is the set of Lagrange multipliers of (48).*

Proof. (i) It follows from Theorems 3.4 and 3.8 that $x_l \to x^*$ for some local minimizer x^* of (50), $\Phi_\rho(x_l) \to \Phi_\rho(x^*)$ $I_l \to I^*$, and

$$x^* \in \text{Arg} \min_{x \in \mathcal{B}_{I^*}} \Phi_\rho(x),$$

where $I_l = I(x_l)$ and $I^* = I(x^*)$. It then follows from Lemma 2.1 (d) that

$$\Phi_\rho(x_l) - \Phi_\rho(x^*) \ge \frac{1}{2L_\rho}\|g(x_l; \rho, I^*)\|^2, \quad l \ge N.$$

Using this inequality and $\Phi_\rho(x_l) \to \Phi_\rho(x^*)$, we thus have $g(x_l; \rho, I^*) \to 0$. Since $I_l = I^*$ for $l \ge N$, we also have $g(x_l; \rho, I_l) \to 0$.

(ii) Let f_I^* be defined in (48). Applying Lemma 2.4 to problem (48), we know that

$$f(x_l) - f_{I(l)}^* \ge -\hat{t}d_{\mathcal{K}^*}(Ax_l - b), \quad \forall l, \tag{61}$$

where $\hat{t}$ is defined above. Let x^* and I^* be defined in the proof of statement (i). We observe that $f_{I^*}^* \ge \Phi_\rho(x^*)$. Using this relation and (61), we have that for sufficiently large l,

$$\begin{aligned}
\Phi_\rho(x_l) - \Phi_\rho(x^*) &= f(x_l) + \tfrac{\rho}{2}[d_{\mathcal{K}^*}(Ax_l - b)]^2 - \Phi_\rho(x^*) \ge f(x_l) - f_{I^*}^* + \tfrac{\rho}{2}[d_{\mathcal{K}^*}(Ax_l - b)]^2 \\
&= f(x_l) - f_{I(l)}^* + \tfrac{\rho}{2}[d_{\mathcal{K}^*}(Ax_l - b)]^2 \ge -\hat{t}d_{\mathcal{K}^*}(Ax_l - b) + \tfrac{\rho}{2}[d_{\mathcal{K}^*}(Ax_l - b)]^2,
\end{aligned}$$

which implies that

$$d_{\mathcal{K}^*}(Ax_l - b) \le \frac{\hat{t}}{\rho} + \sqrt{\frac{\Phi_\rho(x_l) - \Phi_\rho(x^*)}{\rho}}.$$

23

This inequality together with the fact $\lim_{l \to \infty} \Phi_\rho(x_l) = \Phi_\rho(x^*)$ yields statement (ii). ∎

Remark. From Theorem 4.3, we can see that the inner iterations of the above method terminates finitely. ∎

We next establish convergence of the outer iterations of the above variant of the IHT method for (2). In particular, we show that every accumulation point of $\{x^k\}$ is a local minimizer of (2).

Theorem 4.4 *Let $\{x^k\}$ be the sequence generated by the above variant of the IHT method for solving* (2). *Then any accumulation point of $\{x^k\}$ is a local minimizer of* (2).

Proof. Let

$$\tilde{x}^k = \Pi_{\mathcal{B}_{I_k}}(x^k - \frac{1}{L_{\rho_k}}\nabla\Phi_{\rho_k}(x^k)).$$

Since $\{x^k\}$ satisfies (60), it follows from Lemma 2.1 (a) that

$$\nabla\Phi_{\rho_k}(x^k) \in -\mathcal{N}_{\mathcal{B}_{I_k}}(\tilde{x}^k) + \mathcal{U}(\epsilon_k), \tag{62}$$

where $I_k = I(x^k)$. Let x^* be any accumulation point of $\{x^k\}$. Then there exists a subsequence K such that $\{x^k\}_K \to x^*$. By passing to a subsequence if necessary, we can assume that $I_k = I$ for all $k \in K$. Let

$$\mu^k = \rho_k[Ax^k - b - \Pi_{\mathcal{K}^*}(Ax^k - b)].$$

We clearly see that

$$(\mu^k)^T\Pi_{\mathcal{K}^*}(Ax^k - b) = 0. \tag{63}$$

Using (62) and the definitions of Φ_ρ and μ^k, we have

$$\nabla f(x^k) + A^T\mu^k \in -\mathcal{N}_{\mathcal{B}_I}(\tilde{x}^k) + \mathcal{U}(\epsilon_k), \quad \forall k \in \mathcal{K}. \tag{64}$$

By (59), (60) and the definition of $\tilde{x}^k$, one can observe that

$$\|\tilde{x}^k - x^k\| = \frac{1}{L_{\rho_k}}\|g(x^k; \rho_k, I_k)\| \leq \epsilon_k. \tag{65}$$

In addition, notice that $\|\mu^k\| = \rho_k d_{\mathcal{K}^*}(Ax^k - b)$, which together with (60) implies that $\|\mu^k\| \leq t$ for all k. Hence, $\{\mu^k\}$ is bounded. By passing to a subsequence if necessary, we can assume that $\{\mu^k\}_K \to \mu^*$. Using (65) and upon taking limits on both sides of (63) and (64) as $k \in K \to \infty$, we have

$$(\mu^*)^T\Pi_{\mathcal{K}^*}(Ax^* - b) = 0, \quad \nabla f(x^*) + A^T\mu^* \in -\mathcal{N}_{\mathcal{B}_I}(x^*)$$

In addition, since $x_I^k = 0$ for $k \in K$, we know that $x_I^* = 0$. Also, it follows from (60) that $d_{\mathcal{K}^*}(Ax^* - b) = 0$, which implies that $Ax^* - b \in \mathcal{K}^*$. These relations yield

$$x^* \in \text{Arg}\min_{x \in \mathcal{B}_I}\{f(x) : Ax - b \in \mathcal{K}^*\},$$

and hence, x^* is a local minimizer of (2). ∎

5 Concluding remarks

In this paper we studied iterative hard thresholding (IHT) methods for solving l_0 regularized convex cone programming problems. In particular, we first proposed an IHT method and its variant for solving l_0 regularized box constrained convex programming. We showed that the sequence generated by these methods converges to a local minimizer. Also, we established the iteration complexity of the IHT method for finding an ϵ-local-optimal solution. We then proposed a method for solving l_0 regularized convex cone programming by applying the IHT method to its quadratic penalty relaxation and established its iteration complexity for finding an ϵ-approximate local minimizer. Finally, we proposed a variant of this method in which the associated penalty parameter is dynamically updated, and showed that every accumulation point is a local minimizer of the problem.

Some of the methods studied in this paper can be extended to solve some l_0 regularized nonconvex optimization problems. For example, the IHT method and its variant can be applied to problem (12) in which f is nonconvex and ∇f is Lipschitz continuous. In addition, it would be interesting to extend the methods of this paper to solve rank minimization problems and compare them with the methods studied in [14]. This is left as a future research.

Acknowledgment

Appendix

In this appendix we show that if f is smooth strongly convex, there exists $L > L_f$ such that $\alpha > 0$, which is required in Theorem 3.5. In particular, we will show that there exist at most $3n2^{n-1}$ number of L such that $\alpha = 0$; in other words, $\alpha > 0$ for almost all $L > 0$ except at most $3n2^{n-1}$ number of points. Therefore, $\alpha > 0$ is almost sure for a randomly chosen L and an arbitrary $L > L_f$ may be sufficient for practical implementation of the IHT method for solving problem (12).

To this aim, assume that f is smooth strongly convex, which is imposed in Theorem 3.5, and let α be defined in (29). We now show that $\alpha > 0$ for almost all $L > 0$ except at most $3n2^{n-1}$ values. For simplicity, we only consider the case where $-\infty < l_i < 0$ and $0 < u_i < \infty$ for all i. The proof is similar for the other cases. Let $\mathcal{B}_I$, $\Pi_{\mathcal{B}}(\cdot)$ and $s_L(\cdot)$ be defined in (15)-(17), respectively. Given any $I \subseteq \{1, \ldots, n\}$ and $1 \le i \le n$, let

$$h_i(L; I) := [s_L(x^*)]_i^2 - [\Pi_{\mathcal{B}}(s_L(x^*)) - s_L(x^*)]_i^2 - \frac{2\lambda}{L},$$

where $x^* \in \operatorname{Arg\,min}\{f(x) : x \in \mathcal{B}_I\}$. Clearly, $h_i(L; I)$ is well-defined since x^* is unique for each I due to the strongly convexity of f. Notice that α as defined in (29) can be rewritten

as

$$\alpha = \min_{I \subseteq \{1,\dots,n\}} \min_i |h_i(L; I)|. \tag{66}$$

As we will later show, for each $I \subseteq \{1, \dots, n\}$, $h_i(L; I) = 0$ has at most two positive roots L for every $i \in I$ and at most one positive root L for any $i \notin I$. This together with (66) implies that for each $I \subseteq \{1, \dots, n\}$ there exist at most

$$n_I := 2|I| + n - |I| = n + |I|$$

number of L such that $\alpha = 0$, where $|I|$ denotes the size of I. Therefore, the total number of L such that $\alpha = 0$ is at most

$$\sum_{I \subseteq \{1,\dots,n\}} n_I = \sum_{|I|=0}^{n} n_I \binom{n}{|I|} = 3n2^{n-1}.$$

It remains to show that for each $I \subseteq \{1, \dots, n\}$, $h_i(L; I) = 0$ has at most two positive roots L for every $i \in I$ and at most one positive root L for any $i \notin I$. We prove this result by considering all possible cases below. Let $i \in \{1, \dots, n\}$ and $x^* = \arg\min\{f(x) : x \in \mathcal{B}_I\}$.

(1) $[\nabla f(x^*)]_i = 0$: one can see that $[s_L(x^*)]_i = x_i^*$. Hence, $[\Pi_{\mathcal{B}}(s_L(x^*))]_i = x_i^*$ and $h_i(L; I) = (x_i^*)^2 - \frac{2\lambda}{L}$. It follows that $h_i(L; I) = 0$ has at most one positive root L.

(2) $[\nabla f(x^*)]_i \neq 0$: by the first-order optimality condition for $x^* = \arg\min\{f(x) : x \in \mathcal{B}_I\}$, one can observe that if $i \notin I$ and $[\nabla f(x^*)]_i < 0$, then $x_i^* = u_i$; if $i \notin I$ and $[\nabla f(x^*)]_i > 0$, then $x_i^* = l_i$. Hence, this case can be divided into three subcases:

(2a) $i \in I$: it follows that $x_i^* = 0$ and $[s_L(x^*)]_i = -\frac{1}{L}\nabla_i f(x^*)$. Let

$$\theta = \max\left(-\frac{\nabla_i f(x^*)}{l_i}, -\frac{\nabla_i f(x^*)}{u_i}\right).$$

One can observe that

$$[s_L(x^*)]_i \in \begin{cases} [l_i, u_i] & \text{if } L > \theta, \\ (-\infty, l_i] & \text{if } 0 < L \leq \theta \text{ and } \nabla_i f(x^*) > 0, \\ [u_i, \infty) & \text{if } 0 < L \leq \theta \text{ and } \nabla_i f(x^*) < 0. \end{cases}$$

It follows that

$$[\Pi_{\mathcal{B}}(s_L(x^*))]_i = \begin{cases} [s_L(x^*)]_i & \text{if } L > \theta, \\ l_i & \text{if } 0 < L \leq \theta \text{ and } \nabla_i f(x^*) > 0, \\ u_i & \text{if } 0 < L \leq \theta \text{ and } \nabla_i f(x^*) < 0. \end{cases}$$

26

Using this relation and $[s_L(x^*)]_i = -\frac{1}{L}\nabla_i f(x^*)$, we have

$$
h_i(L; I) = \begin{cases}
\frac{1}{L^2}[\nabla_i f(x^*)]^2 - \frac{\lambda}{2L}, & \text{if } L > \theta, \\
-l_i^2 - \frac{2l_i \nabla_i f(x^*)}{L} - \frac{\lambda}{2L}, & \text{if } 0 < L \le \theta \text{ and } \nabla_i f(x^*) > 0, \\
-u_i^2 - \frac{2u_i \nabla_i f(x^*)}{L} - \frac{\lambda}{2L}, & \text{if } 0 < L \le \theta \text{ and } \nabla_i f(x^*) < 0.
\end{cases}
$$

Therefore, one can see that $h_i(L; I) = 0$ has at most two positive roots L.

(2b) $i \notin I$ and $[\nabla f(x^*)]_i < 0$: as mentioned above, $x_i^* = u_i$ holds for this subcase. One can observe that for all $L > 0$,

$$
[s_L(x^*)]_i = x_i^* - \frac{1}{L}\nabla_i f(x^*) > u_i,
$$

and hence $[\Pi_{\mathcal{B}}(s_L(x^*))]_i = u_i$. Using this relation and $x_i^* = u_i$, we have

$$
h_i(L; I) = u_i^2 - \frac{2u_i \nabla_i f(x^*)}{L} - \frac{\lambda}{2L}, \quad \forall L > 0.
$$

Thus, $h_i(L; I) = 0$ has at most one positive root L.

(2c) $i \notin I$ and $[\nabla f(x^*)]_i > 0$: as mentioned above, $x_i^* = l_i$ holds for this subcase. By a similar argument as subcase (2b), one can have

$$
h_i(L; I) = l_i^2 - \frac{2l_i \nabla_i f(x^*)}{L} - \frac{\lambda}{2L}, \quad \forall L > 0.
$$

Hence, $h_i(L; I) = 0$ has at most one positive root L.

Combining all cases above, we obtain the result as desired.

References

[1] S. Bahmani, B. Raj, and P. Boufounos. Greedy sparsity-constrained optimization. *J. Mach. Learn Res.*14: 807–841, 2013.

[2] J. Barzilai and J.M. Borwein. Two point step size gradient methods. *IMA J. Numer. Anal.*, 8:141–148, 1988.

[3] T. Blumensath. Accelerated iterative hard thresholding. *Signal Processing*, 92(3):752–756, 2012.

[4] T. Blumensath. Compressed sensing with nonlinear observations and related nonlinear optimization problems. *IEEE Trans. Inf. Theory*,59(6):3466–3474, 2013.

[5] T. Blumensath and M. E. Davies. Iterative thresholding for sparse approximations. *J. Fourier Anal. Appl.*, 14:629–654, 2008.

[6] T. Blumensath and M. E. Davies. Iterative hard thresholding for compressed sensing. *Appl. Comput. Harmon. Anal.*, 27(3):265–274, 2009.

[7] T. Blumensath and M. E. Davies. Normalized Iterative Hard Thresholding: guaranteed stability and performance. *IEEE J. Sel. Topics Signal Process.*, 4(2):298–309, 2010.

[8] E. Candès, J. Romberg, and T Tao. Robust uncertainty principles: Exact signal reconstruction from highly incomplete frequency information. *IEEE Trans. Inf. Theory*, 52(2):489–509, 2006.

[9] V. Cevher. On accelerated hard thresholding methods for sparse approximation. Conference on Wavelets and Sparsity XIV, San Diego, CA, Proceedings of SPIE, 2011.

[10] A. Dempster. Covariance selection. *Biometrics*, 28:157-175, 1978.

[11] D. Donoho. Compressed sensing. *IEEE Trans. Inf. Theory*, 52(4):1289–1306, 2006.

[12] S. Foucart. Hard thresholding pursuit: an algorithm for compressive sensing. *SIAM J. Numer. Anal.*, 49(6):2543–2563, 2011.

[13] K. K. Herrity, A. C. Gilbert, and J. A. Tropp. Sparse approximation via iterative thresholding. *IEEE International Conference on Acoustics, Speech and Signal Processing*, 2006.

[14] P. Jain, R. Meka, and I. Dhillon. Guaranteed rank minimization via singular value projection. *Neural Information Processing Systems*, 937–945, 2010.

[15] G. Lan and R. D. C. Monteiro. Iteration-complexity of first-order penalty methods for convex programming. *Math. Prog.*, 138:115–139, 2013.

[16] Z. Lu and Y. Zhang. Sparse approximation via penalty decomposition methods. Manuscript, Department of Mathematics, Simon Fraser University, February 2012.

[17] A. Maleki. Coherence analysis of iterative thresholding algorithms. In Proc. of 47th Annual Allerton Conference on Communication, Control, and Computing, 236–243, 2009.

[18] S. Mallat and Z. Zhang. Matching pursuits with time-frequency dictionaries. *IEEE T. Image Process.*, 41(12):3397–3415, 1993.

[19] Y. Nesterov. *Introductory Lectures on Convex Programming: a basic course.* Kluwer Academic Publishers, Massachusetts, 2004.

[20] A. Y. Ng. Feature selection, l_1 vs. l_2 regularization, and rotational invariance. In *Proceedings of the Twenty-First International Conference on Machine learning (ICML)*, 72-85, 2004.

[21] M. Nikolova. Description of the minimizers of least squares regularized with l_0 norm. Report HAL-00723812, CMLA - CNRS ENS Cachan, France, October 2011.

[22] J. Tanner and K. Wei. Normalized iterative hard thresholding for matrix completion. Accepted by *SIAM J. Sci. Comput.*, 2012.

[23] J. A. Tropp. Greed is good: algorithmic results for sparse approximation. *IEEE T. Inform. Theory*, 50(10):2231–2242, 2004.